Spectral Quark Model

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research with Enrique Ruiz Arriola

- ERA + WB, Spectral quark model and low-energy hadron phenomenology, hep-ph/0301202, Phys. Rev. **D** 67 (2003) 074021

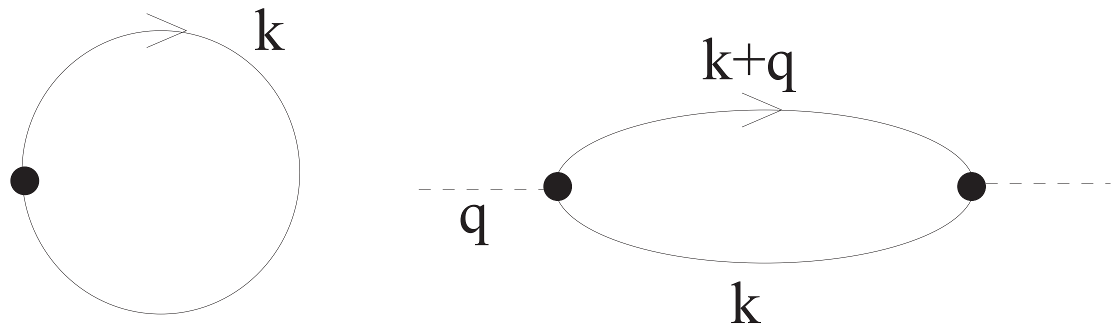
- WB + ERA, Application of chiral quark models to high-energy processes, hep-ph/0410041, Bled 2004, p. 7

[G.V. Efimov and M.A. Ivanov, Int. J. Mod. Phys. A 4 (1989) 2031]

What is a Chiral Quark Model?

Prototype: Nambu-Jona–Lasinio, UV cut-off, interactions $\longrightarrow$ χSB: massive quarks, Goldstone pions, one-loop (leading-N_c)

more than 20 years of vast applications: low-energy hadron spectroscopy and phenomenology (mesons, baryons, pentaquarks, Gasser-Leutwyler coefficients), high-density matter (2SC,CFL), high temperature matter, soft matrix elements for high-energy processes, ...



The momentum running around the loop is cut

$$k < \Lambda$$

Requirements for a quark model

1. Give finite values for hadronic observables

2. Satisfy the electromagnetic and chiral Ward-Takahashi identities, thus reproducing all necessary symmetry requirements

3. Satisfy the anomaly conditions

4. Comply to the QCD factorization property, in the sense that the expansion of a correlator at a large Q is a pure twist-expansion, involving only the inverse powers of Q^2, without the $\log Q^2$ corrections. In other words, scaling violations are due to QCD only, not low-energy physics

5. Have usual dispersion relations

All simultaneously – far from trivial!

Spectral representation

We introduce the spectral regularization of the chiral quark model, based the (generalized) Lehmann representation for the quark propagator,

$$S(p) = \int_C d\omega \frac{\rho(\omega)}{\not{p} - \omega}$$

where $\rho(\omega)$ is the spectral function and C denotes a contour in the complex ω plane chosen in a suitable way. Specific realization will be given later on.

Examples: free theory has $\rho(\omega) = \delta(\omega - m)$, perturbative QCD yields at LO

Non-perturbative?

$$\rho(\omega) = \delta(\omega - m) + \mathrm{sign}(\omega) \frac{\alpha_S C_F}{4\pi} \frac{1-\xi}{\omega} \theta(\omega^2 - m^2)$$

Quark condensate

$$\langle \bar{q} q \rangle \equiv -iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} S(p) = -4iN_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega}{p^2 - \omega^2}$$

The integral over p is quadratically divergent, which requires the use of an auxiliary regularization, *removed* at the end

$$\langle \bar{q} q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \omega \rho(\omega) \left[2\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{4\Lambda^2}\right) + \omega^2 + \mathcal{O}(1/\Lambda) \right]$$

The finiteness of the result at $\Lambda \to \infty$ requires the conditions

$$\int d\omega \omega \rho(\omega) = 0, \quad \int d\omega \omega^3 \rho(\omega) = 0$$

and thus

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \log(\omega^2) \omega^3 \rho(\omega).$$

The spectral condition allowed for rewriting $\log(\omega^2/\Lambda^2)$ as $\log(\omega^2)$, hence no scale dependence (no "dimensional transmutation") is present in the final expression.

With the accepted value of

$$\langle \bar{q}q \rangle = \simeq -(243 \text{ MeV})^3$$

we infer the value of the third log-moment. The negative sign of the quark condensate shows that

$$\int d\omega \log(\omega^2) \omega^3 \rho(\omega) > 0.$$

Vacuum energy density

$$\langle \theta^{\mu\nu} \rangle = -iN_c N_f \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \times$$
$$\mathrm{Tr} \frac{1}{\not{p} - \omega} \left[\frac{1}{2} (\gamma^\mu p^\nu + \gamma^\nu p^\mu) - g^{\mu\nu} (\not{p} - \omega) \right] = B g^{\mu\nu} + \langle \theta^{\mu\nu} \rangle_0,$$

where $\langle \theta^{\mu\nu} \rangle_0$ is the energy-momentum tensor for the free theory, evaluated with $\rho(\omega) = \delta(\omega)$, and B (bag constant) is the vacuum energy density:

$$B = -iN_c N_f \int_C d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega^2}{p^2 - \omega^2},$$

The conditions that have to be fulfilled for B to be finite are

$$\rho_2 = 0, \quad \rho_4 = 0$$

Then

$$B = -\frac{N_c N_f}{16\pi^2}\rho_4' \equiv -\frac{N_c N_f}{16\pi^2}\int_C d\omega \log(\omega^2)\omega^4 \rho(\omega)$$

According to the most recent QCD sum rules analysis

$$B = -\frac{9}{32}\langle\frac{\alpha}{\pi}G^2\rangle = -(224^{+35}_{-70}\mathrm{MeV})^4$$

The negative sign of B enforces

$$\rho_4' > 0$$

[one can go on for other observables] $\longrightarrow$

Spectral moments

Normalization (at $p \to \infty$ we have $S(p) \to 1/\not{p}$):

$$\rho_0 \quad \equiv \int_C d\omega \rho(w) = 1,$$

Finiteness of observables:

$$\rho_n \quad \equiv \int_C d\omega \omega^n \rho(\omega) = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Observables are given by inverse moments

$$\rho_{-k} \quad \equiv \int_C d\omega \omega^{-k} \rho(\omega), \quad \text{for } k = 1, 2, 3, \ldots$$

as well as by the "log moments",

$$\rho'_n \equiv \int_C d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n = 2, 3, 4, \ldots$$

Such a $\rho(\omega)$ and C exist!

Effective action

The effective action of the model is (in the chiral limit)

$$S_{\rm eff} = -iN_c \int_C d\omega \rho(\omega) \mathrm{Tr} \log(i\not{D} - \omega U^{\gamma_5})$$

with D denoting the covariant derivative and the chiral fields entering $U^{\gamma_5} = \exp(-i\gamma_5 \tau \cdot \theta)$

Sightseeing tour $\longrightarrow$

Elecromagnetic vacuum polarization

$$
\begin{aligned}
i\Pi^{\mu a,\nu b}_{VV}(q) &= \delta^{ab}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\bar{\Pi}_{VV}(q) = \\
&= -N_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - \not q - \omega}\gamma_\mu \frac{\lambda_a}{2}\frac{i}{\not p - \omega}\gamma_\nu \frac{\lambda_b}{2}\right]
\end{aligned}
$$

transverse!

$$
\begin{aligned}
\bar{\Pi}_{VV}(q) &= \frac{N_c}{3}\int_C d\omega \rho(\omega)\left\{-2\omega^2[I(q^2,\omega) - I(0,\omega)] + q^2[\frac{1}{3} - I(q^2,\omega)]\right\} \\
I(q^2,\omega) &= -\frac{1}{(4\pi)^2}\int_0^1 dx \log\left[\omega^2 + x(1-x)q^2\right]
\end{aligned}
$$

The usual twice-subtracted dispersion relation holds:

$$
\bar{\Pi}_V(q^2) = \frac{q^4}{\pi}\int_0^\infty \frac{dt}{t^2}\frac{\mathrm{Im}\bar{\Pi}_V(t)}{t - q^2 - i0^+}
$$

This is in contrast to (non-local) quark models formulated in or the popular proper-time regularization

$e^+e^- \to$ hadrons

At large s we find

proportional to $\mathrm{Im}\bar{\Pi}_V$

$$\sigma(e^+e^- \to \mathrm{hadrons}) \to \frac{4\pi\alpha_{\mathrm{QED}}^2}{3s}\left(\sum_i e_i^2\right)\int d\omega\rho(\omega),$$

where e_i is the electric charge of the quark of species i. This is the proper asymptotic QCD result, provided

$$\int d\omega\rho(\omega) = 1$$

Pion weak decay

The pion weak-decay constant, defined as

$$\langle 0 \,|\, J_A^{\mu a}(x) |\, \pi_b(q) \rangle = i f_\pi q_\mu \delta_{a,b} e^{iq\cdot x},$$

can be computed from the axial-axial correlation function. The result is

$$f_\pi^2 \;=\; 4N_c \int d\omega \rho(\omega) \omega^2 I(0,\omega)$$

A finite value for f_π requires the condition $\rho_2 = 0$. Then

$$f_\pi^2 = -\frac{N_c}{4\pi^2} \int d\omega \log(\omega^2) \omega^2 \rho(\omega) \equiv -\frac{N_c}{4\pi^2} \rho_2'$$

The value $f_\pi = 93\text{MeV}$ determines ρ_2'. The sign is

$$\rho_2' < 0$$

Pion electromagnetic form factor

The electromagnetic form factor for a positively charged pion, $\pi^+ = u\bar{d}$, is defined as

$$\langle \pi^+(p')|J_\mu^{\rm em}(0)|\pi^+(p)\rangle \quad = \quad (p^\mu + p'^\mu) e F_\pi^{\rm em}(q^2)$$

For on-shell massless pions the electromagnetic form factor reads

$$F_\pi^{em}(q^2) = \frac{4N_c}{f_\pi^2} \int dw \rho(\omega)\omega^2 I(q^2,\omega)$$

The low-momentum expansion is $F_\pi^{em}(0) = 1$

$$F_\pi^{em}(q^2) = 1 + \frac{1}{4\pi^2 f_\pi^2}\left(\frac{q^2\rho_0}{6} + \frac{q^4\rho_{-2}}{60} + \frac{q^6\rho_{-4}}{240} + \dots\right)$$

The mean squared radius reads

$$\langle r_\pi^2 \rangle = 6 \frac{dF}{dq^2}|_{q^2=0} = \frac{N_c}{4\pi^2 f_\pi^2} \int d\omega \rho(\omega) = \frac{N_c}{4\pi^2 f_\pi^2},$$

which coincides with the unregularized-quark-loop result. The numerical value is

$$\langle r^2 \rangle_\pi^{\mathrm{em}} \Big|_{\mathrm{th}} = 0.34 \mathrm{fm}^2, \qquad \langle r^2 \rangle_\pi^{\mathrm{em}} \Big|_{\mathrm{exp}} = 0.44 \mathrm{fm}^2,$$

which is a reasonable agreement (χPT corrections).

The knowledge of the pion electromagnetic form factor allows to determine the even negative moments of $\rho(\omega)$.

Twist expansion and spectral conditions

In the limit of large momentum

$$F_\pi^{em}(q^2) \sim \frac{N_c}{4\pi^2 f_\pi^2}\int d\omega\rho(\omega)\omega^2\{2-\frac{1}{\epsilon}-\log(q^2)+ \frac{2\omega^2}{q^2}\left[\log(-q^2/\omega^2)+1\right]+\frac{2\omega^4}{q^4}\left[\log(-q^2/\omega^2)-\frac{1}{2}\right]\ldots\}$$

With help of the spectral conditions for $n=2,4,6,\ldots$ we get

$$F_\pi^{em}(q^2) \sim -\frac{N_c}{4\pi^2 f_\pi^2}\left[\frac{2\rho'_4}{q^2}+\frac{2\rho'_6}{q^4}+\frac{4\rho'_8}{q^6}+\ldots\right]$$

All spectral conditions needed!

The imposition of the spectral conditions removed all the logs from the expansion, leaving a pure expansion in inverse powers of q^2!

Anomalous decay $\pi^0 \to \gamma\gamma$

$$\Gamma^{\mu\nu}_{\pi^0\gamma\gamma}(q_1, q_2) = \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{\pi\gamma\gamma}(q_1, q_2) = -N_c \int d\omega \rho(w) \int \frac{d^4k}{(2\pi)^4} \times$$

$$\mathrm{Tr}\left[-\frac{\omega}{f_\pi} \gamma_5 \tau_3 \frac{i}{\not k - \not q_2 - \omega} i\hat{Q}\gamma^\mu \frac{i}{\not k - \omega} i\hat{Q}\gamma^\nu \frac{i}{\not k - \not q_1 - \omega} \right] + \mathrm{crossed}$$

where $\hat{Q} = \frac{1}{2}N_c + \frac{\tau_3}{2}$. We find (as if no regulator were present)

Blin, Hiller & Schaden

$$\begin{aligned} F_{\pi\gamma\gamma}(0,0) &= \frac{8}{f_\pi} \int d\omega \rho(\omega)\omega^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2-\omega^2)^3} \\ &= \frac{1}{4\pi^2 f_\pi} \int d\omega \rho(\omega) = \frac{1}{4\pi^2 f_\pi} \end{aligned}$$

which coincides with the standard QCD result!

Pion-photon transition form factor

For two off-shell photons with momenta q_1 and q_2 one defines the asymmetry, A, and the total virtuality, Q^2:

$$A = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \qquad -1 \le A \le 1, \;\; Q^2 = -(q_1^2 + q_2^2)$$

At the soft pion point we find the expansion

$$F_{\pi\gamma\gamma}(Q^2, A) = -\frac{1}{2\pi^2 f_\pi} \int_0^1 dx \left[\frac{2\rho_2'}{Q^2(1 - A^2(2x-1)^2)} + \dots \right]$$

We can confront this with the standard twist decomposition of the pion transition form factor ,

$$F_{\gamma\gamma\pi}(Q^2, A) = J^{(2)}(A)\frac{1}{Q^2} + J^{(4)}(A)\frac{1}{Q^4} + \dots,$$

which yields

$$J^{(2)}(A) \quad = \quad \frac{4 f_\pi}{N_c} \int_0^1 dx \frac{\varphi(x; Q_0)}{1 - (2x-1)^2 A^2}$$

with the leading-twist pion distribution amplitude $\varphi(x; Q_0) = 1$

Brodsky-Lepage, Praszałowicz-Rostworowski, Dorokhov

Pion light-cone wave function

The approach yields the following light-cone pion wave function:

$$\Psi(x,k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}\int_C d\omega \rho(\omega)\frac{\omega^2}{k_\perp^2+\omega^2}\theta(x)\theta(1-x)$$

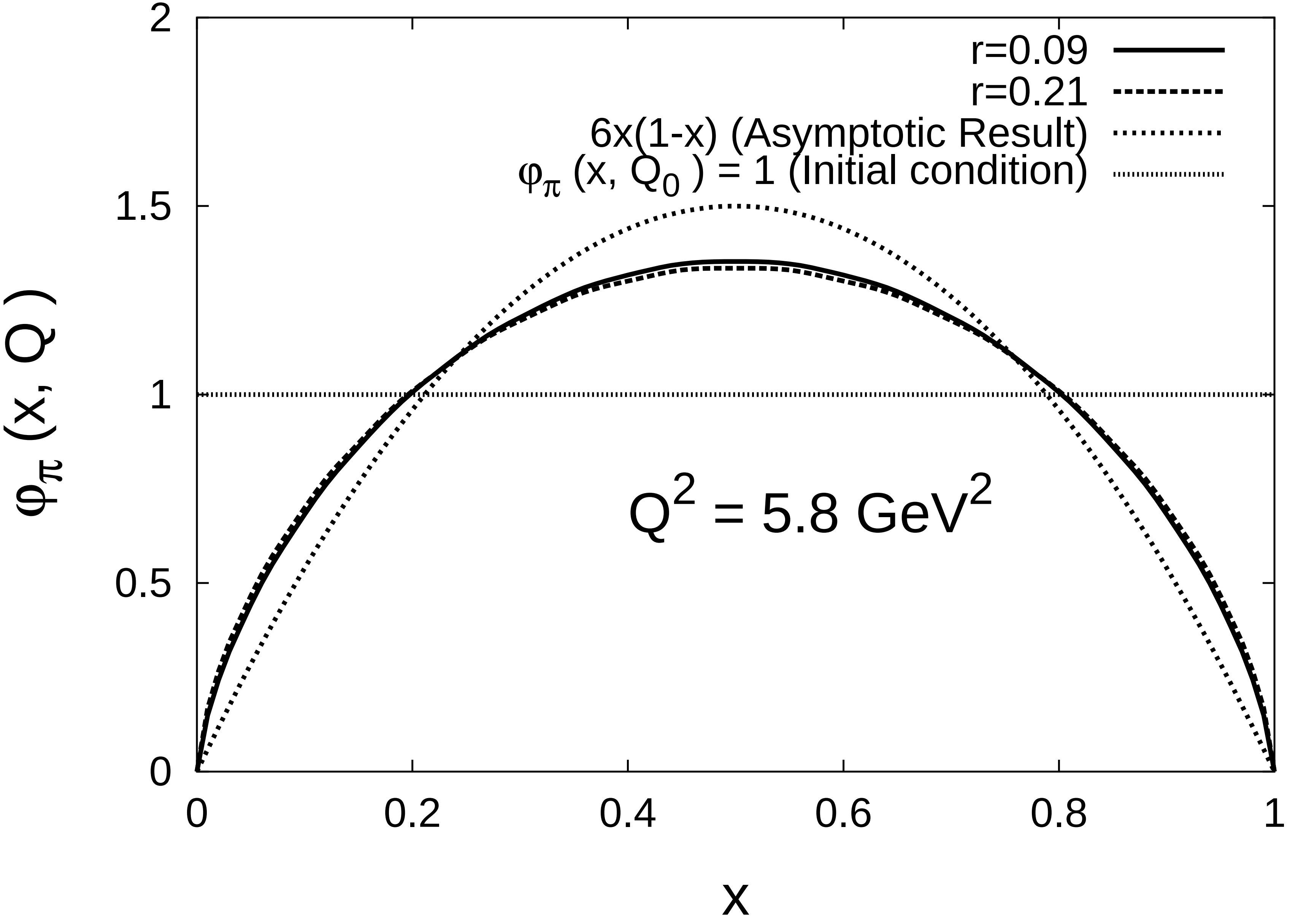
(at the low-energy scale of the model, Q_0). It has correct support and normalization, since $\int d^2k_\perp \Psi(x,k_\perp) = \varphi(x) = 1$. At $k_\perp = 0$ it satisfies the condition:

$$\Psi(x,0) = \frac{N_c}{\pi f_\pi}F_{\pi\gamma\gamma}(0,0) = \frac{N_c}{4\pi^3 f_\pi^2}$$

In QCD one has a similar relation holding for quantities integrated over x

QCD evolution $\longrightarrow$

[details in ERA+WB, Phys. Rev. **D66**(2002)094016]


r=0.09
r=0.21
6x(1-x) (Asymptotic Result)
φπ (x, Q0) = 1 (Initial condition)
Q2 = 5.8 GeV2
φπ (x, Q)
x
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2

The $\gamma \to \pi^+\pi^0\pi^-$ decay

This is an example of a low-energy process involving a quark box diagram, which similarly to the neutral pion decay is related to the QCD anomaly in the soft pion limit. The amplitude for $\gamma(q,e) \to \pi^+(p_1)\pi^0(p_2)\pi^-(p_3)$ is

$$T_{\gamma(q,\varepsilon)\to\pi^+(p_1)\pi^0(p_2)\pi^-(p_3)} \equiv F(p_1,p_2,p_3)\varepsilon_{\alpha\beta\sigma\tau}e^\alpha p_1^\beta p_2^\sigma p_3^\tau.$$

In the limit of all momenta going to zero we get (immediately!), with the spectral normalization condition,

$$F(0,0,0) = \frac{1}{4\pi^2 f_\pi^3}\int d\omega\rho(\omega) = \frac{1}{4\pi^2 f_\pi^3}$$

which is the correct result!

Pion structure function

We take π^+ for definiteness and get

$$u_\pi(x) = \bar{d}_\pi(1-x) = \theta(x)\theta(1-x),$$

Davidson & Arriola in NJL

The $k_\perp$-unintegrated parton distribution is equal to

$$q(x,k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}\int d\omega\rho(\omega)\frac{\omega^2}{k_\perp^2+\omega^2}\theta(x)\theta(1-x),$$

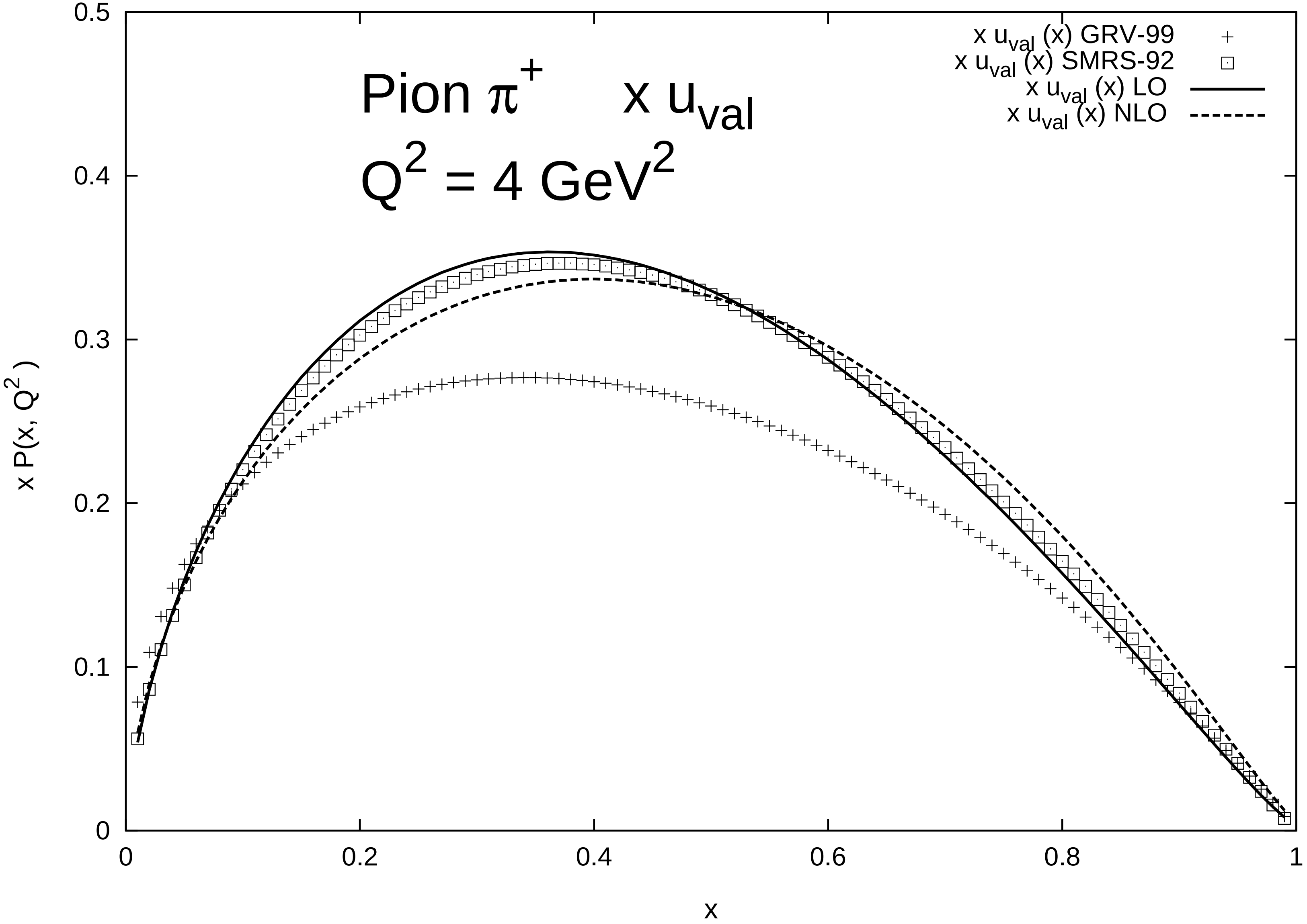
hence (at Q_0) one has an interesting relation

$$q(x,k_\perp) = \bar{q}(1-x,k_\perp) = \Psi(x,k_\perp), \quad q(x) = \varphi(x).$$

The first moment of the PDF is responsible for the momentum sum rule. We find

$$\int_0^1 dx\, xq(x) = \int_0^1 dx\, x\bar{q}(x) = \frac{1}{2}.$$

quarks carry all momentum


Pion π^+ x u_{val}
$Q^2 = 4$ GeV^2
x u_{val} (x) GRV-99
x u_{val} (x) SMRS-92
x u_{val} (x) LO
x u_{val} (x) NLO
x P(x, Q^2)
x
0
0.1
0.2
0.3
0.4
0.5
0.2
0.4
0.6
0.8
1

Résumé

Spectral condition	Physical significance
normalization	
$\rho_0 = 1$	proper normalization of the quark propagator
	preservation of anomalies
	proper normalization of the pion distribution amplitude
	proper normalization of the pion structure function
	reproduction of the large-N_c quark-model values of the Gasser-Leutwyler coefficients
	relation $M_V^2 = 24\pi^2 f_\pi^2/N_c$ in the VMD model
positive moments	
$\rho_1 = 0$	finiteness of the quark condensate, $\langle \bar{q}q \rangle$
	vanishing quark mass at asymptotic Euclidean momenta,
$\rho_2 = 0$	finiteness of the vacuum energy density, B
	finiteness of the pion decay constant, f_π
$\rho_3 = 0$	finiteness of the quark condensate, $\langle \bar{q}q \rangle$
$\rho_4 = 0$	finiteness of the vacuum energy density, B
$\rho_n = 0, \ n = 2, 4 \ldots$	absence of logs in the twist expansion of vector amplitudes
$\rho_n = 0, \ n = 5, 7 \ldots$	finiteness of nonlocal quark condensates, $\langle \bar{q}(\partial^2)^{(n-3)/2} q \rangle$
	absence of logs the twist expansion of the scalar pion form factor

Spectral condition	Physical significance
negative moments	
$\rho_{-2} > 0$	positive quark wave-function normalization at vanishing momentum
$\rho_{-1}/\rho_{-2} > 0$	positive value of the quark mass at vanishing momentum, $M(0) > 0$
ρ_{-n}	low-momentum expansion of correlators
log-moments	
$\rho'_2 < 0$	$f_\pi^2 = -N_c/(4\pi^2)\rho'_2$
$\rho'_3 > 0$	negative value of the quark condensate, $\langle \bar{q}q \rangle = -N_c/(4\pi^2)\rho'_3$
$\rho'_4 > 0$	negative value of the vacuum energy density, $B = -N_c/(4\pi^2)\rho'_4$
$\rho'_5 < 0$	positive value of the squared vacuum virtuality of the quark, $\lambda_q^2 = -\rho'_5/\rho'_3$
ρ'_n	high-momentum (twist) expansion of correlators

Vector-meson dominance in SQM

Now we construct explicitly an example of $\rho(\omega)$ doing the job. Vector-meson dominance (VMD) of the pion form factor is assumed:

$$F_V^{\mathrm{exp}}(t) = \frac{M_V^2}{M_V^2 + t}.$$

with $M_V = m_\rho$. In our approach

$$F_V^{\mathrm{SQM}}(t) = \frac{N_c}{4\pi^2 f_\pi^2} \sum_{n=1}^{\infty} \rho_{2-2n} \frac{2^{-2n-1}\sqrt{\pi}\Gamma(n+1)}{n\Gamma(n+3/2)} (-t)^n .$$

Comparison yields

$$\rho_{2-2k} = \frac{2^{2k+3}\pi^{3/2} f_\pi^2}{N_c M_V^{2k}} \frac{k\,\Gamma(k+3/2)}{\Gamma(k+1)}, k = 1, 2, 3, \ldots$$

In particular, the normalization condition, $\rho_0 = 1$, yields

$$M_V^2 = \frac{24\pi^2 f_\pi^2}{N_c}$$

This relation is usually obtained when matching chiral quark models to VMD, yielding $M_V = 826\ \mathrm{MeV}$ with $f_\pi = 93$ MeV, and $M_V = 764\ \mathrm{MeV}$ with $f_\pi = 86$ MeV (in the chiral limit).

The positive even moments are obtained by analytic continuation in the index n. They fulfill the spectral conditions of vanishing of the positive moments since $\Gamma(n)$ has single poles at non-positive integers, $n = 0, -1, -2, \ldots$ Miracle!

$$\rho_{2n} = 0, \qquad n = 1, 2, 3 \ldots$$

For the even log-moments we have

$$\rho'_{2n} \;=\; \left(-\frac{M_V^2}{4}\right)^n \frac{\Gamma(n)\,\Gamma\left(\frac{5}{2}-n\right)}{\Gamma(\frac{5}{2})}, \qquad n = 1, 2, 3 \ldots$$

The first few values are

$$\rho'_2 = -\frac{4f^2\pi^2}{N_c}, \quad \rho'_4 = \frac{2f^2M_V^2\pi^2}{N_c}$$

We may write the following interesting relation coming out from VMD and the spectral approach:

$$B \;=\; -\frac{9\pi^2 f_\pi^4}{N_c} = -\frac{N_c M_V^4}{64\pi^2} = -(202 - 217 \text{ MeV})^4$$

which agrees within errors with the QCD SR estimate.

The inverse problem

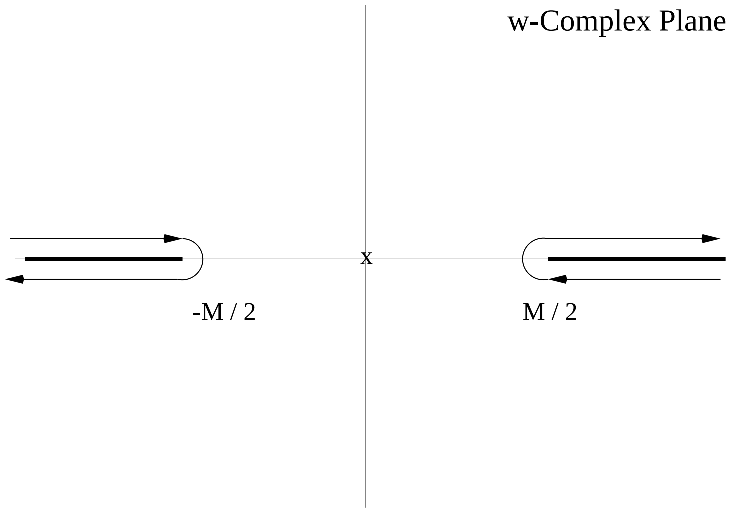
The mathematical problem is now to invert the formula

$$\rho_{2n} = \int_C d\omega \omega^{2n} \rho_V(\omega)$$

The solution is given by the following surprisingly simple function

$$\rho_V(\omega) = \frac{1}{2\pi i} \frac{1}{\omega} \frac{1}{(1 - 4\omega^2/M_V^2)^{5/2}}.$$

The function $\rho_V(\omega)$ has a single pole at the origin and branch cuts starting at $\pm$ half the meson mass, $\omega = \pm M_V/2$.



contour C

Scalar spectral function

$\rho = \rho_V + \rho_S$

For the case of the scalar spectral function (controling the odd moments) we proceed heuristically, by proposing

$$\rho_S(\omega) = \frac{1}{2\pi i} \frac{12\rho_3'}{M_S^4(1 - 4\omega^2/M_S^2)^{5/2}}$$

where the normalization is chosen in such a way that $\rho_3' = -4\pi^2 \langle \bar{q} q \rangle / N_c$.

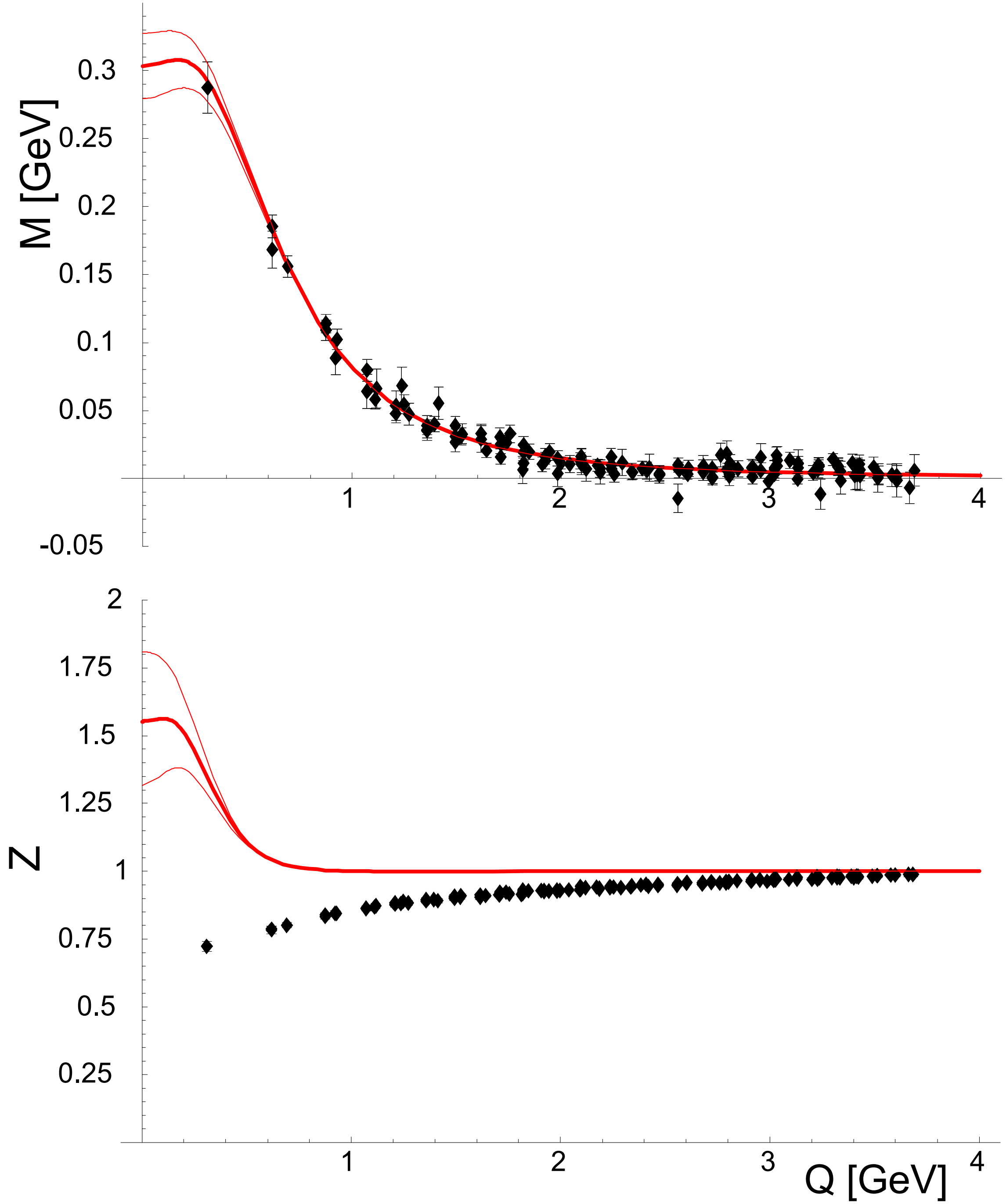
The analytic structure of $\rho_S(\omega)$ is similar to the case of $\rho_V(\omega)$, except for the absence of the pole at $\omega = 0$.

The quark propagator

$$
\begin{aligned}
S(p) &= A(p)\not{p} + B(p) = Z(p)\frac{\not{p} + M(p)}{p^2 - M^2(p)} \\
A(p^2) &\equiv \int_C d\omega \frac{\rho_V(\omega)}{p^2 - \omega^2} = \frac{1}{p^2}\left[1 - \frac{1}{(1-4p^2/M_V^2)^{5/2}}\right] = -\frac{10}{M_V^2} - \frac{70p^2}{M_V^4} + \ldots \\
B(p^2) &\equiv \int_C d\omega \frac{\omega \rho_S(\omega)}{p^2 - \omega^2} = \frac{48\pi^2\langle\bar{q}q\rangle}{M_S^4 N_c (1-4p^2/M_S^2)^{5/2}}
\end{aligned}
$$

No poles in the whole complex plane! Only branch cuts starting at $p^2 = 4M_{V/S}^2$ (obvious, since poles would lead to cuts in the pion form factor)

The absence of poles is sometimes called "analytic confinement"


M [GeV]
0.3
0.25
0.2
0.15
0.1
0.05
-0.05
1
2
3
4
Z
2
1.75
1.5
1.25
1
0.75
0.5
0.25
1
2
3
4
Q [GeV]

$M(Q^2)$ decreases as $1/Q^3$ at large Euclidean momenta, which is favored by the recent lattice calculations. The χ^2 fit results in the following optimum values,

$$\begin{aligned} M_0 &= 303 \pm 24 \text{ MeV}, \\ M_S &= 970 \pm 21 \text{ MeV}, \end{aligned}$$

with the optimum value of χ^2 per degree of freedom equal to 0.72. The corresponding value of the quark condensate is

$$\langle \bar{q} q \rangle = -(243.0^{+0.1}_{-0.8} \text{ MeV})^3.$$

Other predictions

Pion light-cone wave function:

$$\Psi(x,k_\perp) = \frac{3M_V^3}{16\pi(k_\perp^2 + M_V^2/4)^{5/2}}\theta(x)\theta(1-x)$$

The average transverse momentum squared is equal to

$$\langle k_\perp^2 \rangle \equiv \int d^2k_\perp \, k_\perp^2 \Psi(x,k_\perp) = \frac{M_V^2}{2}$$

which numerically gives $\langle k_\perp^2 \rangle = (544 \text{ MeV})^2$ (at Q_0).

Unintegrated PDF:

$$q(x,k_\perp) = \overline{q}(1-x,k_\perp) = \frac{3M_V^3}{16\pi(k_\perp^2 + M_V^2/4)^{5/2}}\theta(x)\theta(1-x)$$

Quark propagator in the coordinate representation:

$$A(x) = \frac{48 + 24M_V\sqrt{-x^2} - 6M_V^2 x^2 + M_V^3(-x^2)^{3/2}}{96\pi^2 x^4}\exp(-M_V\sqrt{-x^2}/2)$$

$$B(x) = \langle\overline{q}q\rangle/(4N_c)\exp(-M_S\sqrt{-x^2}/2)$$

Final remarks

1. What has been done? Spectral representation, one loop. All seems to work in the pion sector: symmetries, anomalies, normalization guaranteed. Dynamics encoded in the spectral inverse and log moments

2. The method is very simple and predictive, having lots of applications

3. No $\log(Q^2)$ generated by the low-energy model (factorization property). This allows for calculations of matrix element for the high-energy processes (DAs, PDFs, pion transition form factor, ...)

4. Remark: need for QCD evolution of all quantities in order to pass from the quark model scale Q_0 to a high-energy scale Q.

5. Interesting particular realization: VMD SQM

6. The form of the quark propagator: no poles, only cuts

7. More to come: photon and ρ-meson light-cone wave functions, sub-leading twist pion LCWFs

Backup slides

Coupling of electroweak currents

In QCD, the vector and axial currents are: reminder

$$J_V^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \frac{\lambda_a}{2} q(x), \quad J_A^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x)$$

CVC and PCAC:

$$\partial_\mu J_V^{\mu,a}(x) = 0, \quad \partial_\mu J_A^{\mu,a}(x) = \bar{q}(x)\hat{M}_0 i\gamma_5 \frac{\lambda_a}{2} q(x)$$

This implies Ward-Takahashi identities, based on

$$\left[J_V^{0,a}(x), q(x')\right]_{x_0=x_0'} = -\frac{\lambda_a}{2} q(x)\delta(\vec{x}-\vec{x}')$$

$$\left[J_A^{0,a}(x), q(x')\right]_{x_0=x_0'} = -\gamma_5 \frac{\lambda_a}{2} q(x)\delta(\vec{x}-\vec{x}')$$

A number of results are then obtained essentially for free: pions arise as Goldstone bosons with standard current-algebra properties, at high energies parton-model features, such as scaling or the spin-1/2 nature of hadronic constituents, are recovered

Vertices with one current

The vector and axial unamputated vertex functions are:

$$\Lambda^{\mu,a}_{V,A}(p',p) = \int d^4x d^4x' \langle 0|T\left\{J^{\mu,a}_{V,A}(0)q(x')\bar{q}(x)\right\}|0\rangle e^{ip'\cdot x' - ip\cdot x}$$

WTIs:

$$(p'-p)_\mu \Lambda^{\mu,a}_V(p',p) = S(p')\frac{\lambda_a}{2} - \frac{\lambda_a}{2}S(p)$$

$$(p'-p)_\mu \Lambda^{\mu,a}_A(p',p) = S(p')\frac{\lambda_a}{2}\gamma_5 + \gamma_5\frac{\lambda_a}{2}S(p)$$

Solution

$$\Lambda^{\mu,a}_V(p',p) = \int d\omega \rho(\omega)\frac{i}{\not{p}'-\omega}\gamma^\mu\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}$$

$$\Lambda^{\mu,a}_A(p',p) = \int d\omega \rho(\omega)\frac{i}{\not{p}'-\omega}\left(\gamma^\mu - \frac{2\omega q^\mu}{q^2}\right)\gamma_5\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}$$

Delburgo & West: gauge technique

Pion-quark coupling

Near the pion pole $(q^2 = 0)$ we get

$$\Lambda_A^{\mu,a}(p+q,p) \to -\frac{q^\mu}{q^2}\Lambda_\pi^a(p+q,p),$$

where

$$\Lambda_\pi^a(p+q,p) = \int d\omega \rho(\omega) \frac{i}{\not p + \not q - \omega} \frac{\omega}{f_\pi} \gamma_5 \lambda_a \frac{i}{\not p - \omega}$$

We recognize in our formulation the Goldberger-Treiman relation for quarks:

$$g_\pi(\omega) = \frac{\omega}{f_\pi}$$

“Transverse ambiguity”

The above ansätze fulfill the WTI's. They are determined up to *transverse pieces*.

This ambiguity appears in all effective models. Current conservation fixes only the longitudinal pieces. Example:

$$j_\mu = \bar{\psi} \left(f_1 \gamma_\mu + i f_2 \sigma_{\mu\nu} q^\nu \right) \psi$$

The condition $q^\mu j_\mu = 0$ does not constrain the f_2-term, since $\sigma_{\mu\nu} q^\nu q^\mu = 0$ from antisymmetry.

QCD evolution of PDA

All results of the effective, low-energy model, refer to a soft energy scale, Q_0. In order to compare to experimental results, obtained at large scales, Q, the QCD evolution must be performed. Initial condition:

$$\varphi(x;Q_0) = \theta(x)\theta(1-x).$$

The evolved distribution amplitude reads

$$\begin{aligned}
\varphi(x;Q) &= 6x(1-x)\sum_{n=0}^{\infty} C_n^{3/2}(2x-1)a_n(Q) \\
a_n(Q) &= \frac{2}{3}\frac{2n+3}{(n+1)(n+2)}\left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)}\right)^{\gamma_n^{(0)}/(2\beta_0)}
\end{aligned}$$

where $C_n^{3/2}$ are the Gegenbauer polynomials, $\gamma_n^{(0)}$ are appropriate anomalous dimensions, and $\beta_0 = 9$.

Results extracted from the experimental data of CLEO provide

$a_2(2.4\mathrm{GeV}) = 0.12 \pm 0.03$, which we use to fix

$$\alpha(Q = 2.4\mathrm{GeV})/\alpha(Q_0) = 0.15 \pm 0.06$$

At LO this correspondsto $Q_0 = 322 \pm 45$ MeV

Now we can predict

$$\begin{aligned} a_4(2.4\mathrm{GeV}) &= 0.06 \pm 0.02 \;\; (\mathrm{exp}: -0.14 \pm 0.03 \mp 0.09) \\ a_6(2.4\mathrm{GeV}) &= 0.02 \pm 0.01 \end{aligned}$$

Encouraging, with leading-twist and LO QCD evolution!

QCD evolution of PDF

The QCD evolution of the constant PDF has been treated in detail by Davidson & ERA at LO and NLO. In particular, the non-singlet contribution to the energy-momentum tensor evolves as

$$\frac{\int dx\, xq(x,Q)}{\int dx\, xq(x,Q_0)} = \left(\frac{\alpha(Q)}{\alpha(Q_0)}\right)^{\gamma_1^{(0)}/(2\beta_0)},$$

In has been found that at $Q^2 = 4\mathrm{GeV}^2$ the valence quarks carry $47 \pm 0.02\%$ of the total momentum fraction in the pion. Downward LO evolution yields that at the scale

$$Q_0 = 313^{+20}_{-10}\mathrm{MeV}$$

the quarks carry 100% of the momentum. The agreement of the evolved PDF with the SMRS data analysis is impressive