From analyticity to confinement: the spectral quark model

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Coimbra, 7 May 2003

- Spectral quark model and low-energy hadron phenomenology, hep-ph/0301202, Phys. Rev. **D67** (2003) 074021

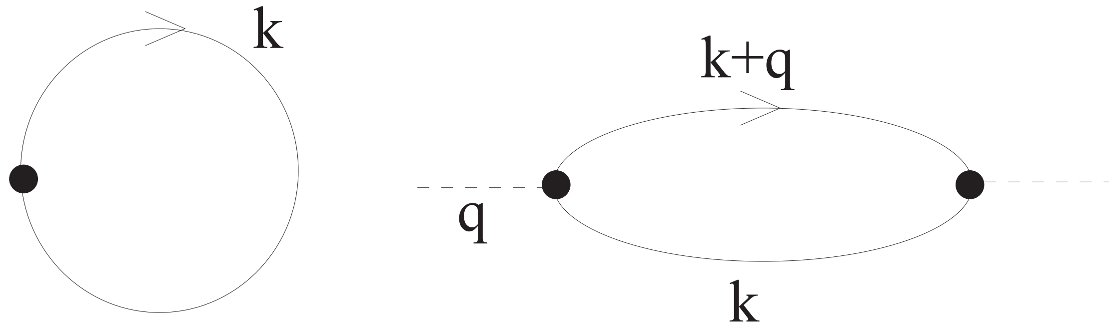
- Pion light-cone wave function and pion distribution amplitude in the Nambu–Jona-Lasinio model, hep-ph/0207266, Phys. Rev. **D66** (2002) 094016

What is a Chiral Quark Model?

Prototype: Nambu-Jona–Lasinio, UV cut-off, interactions
$\to \chi$SB: massive quarks, Goldstone pions

One-loop (leading-N_c)

42 years since invention, 20 years of vast applications:
low-energy hadron phenomenology (pions, nucleons, quark matter, ...)



The momentum running around the loop is cut, $k < \Lambda$

This is not what we are going to do!

Requirements

1. Give finite values for hadronic observables

2. Satisfy the Ward-Takahashi identities, thus reproducing all necessary symmetry requirements

3. Satisfy the anomaly conditions

4. Comply to the QCD factorization property, in the sense that the expansion of a correlator at a large Q is a pure twist-expansion, involving only the inverse powers of Q^2, without the $\log Q^2$ corrections

5. Have the usual dispersion relations

All simultaneously – far from trivial!

Spectral representation

We introduce a novel approach, the spectral regularization of the chiral quark model, based on the Lehmann representation

$$S(p) = \int_C d\omega \frac{\rho(\omega)}{\not{p} - \omega}$$

$\rho(\omega)$ – the spectral function, C – a suitable contour in the complex ω plane

Examples: free theory has $\rho(\omega) = \delta(\omega - m)$, perturbative QCD yields at LO

Non-perturbative?

$$\rho(\omega) = \delta(\omega - m) + \mathrm{sign}(\omega)\frac{\alpha_S C_F}{4\pi}\frac{1-\xi}{\omega}\theta(\omega^2 - m^2)$$

Quark condensate

$$\langle \bar{q} q \rangle \equiv -iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} S(p) = -4iN_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega}{p^2 - \omega^2}$$

The integral over p is quadratically divergent, which requires the use of an auxiliary regularization, *removed* at the end

$$\langle \bar{q} q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \omega \rho(\omega) \left[2\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{4\Lambda^2} \right) + \omega^2 + \mathcal{O}(1/\Lambda) \right]$$

The finiteness of the result at $\Lambda \to \infty$ requires the conditions

The Arriola conditions

$$\int d\omega \omega \rho(\omega) = 0, \quad \int d\omega \omega^3 \rho(\omega) = 0$$

and thus

$$\langle \bar{q} q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \log(\omega^2) \omega^3 \rho(\omega)$$

The spectral condition allowed us to rewrite $\log(\omega^2/\Lambda^2)$ as $\log(\omega^2)$, hence no scale dependence (no "dimensional transmutation") is present in the final expression

With the accepted value of

$$\langle \bar{q} q \rangle = \simeq -(243 \text{ MeV})^3$$

we infer the value of the third log-moment. The negative sign of the quark condensate shows that

$$\int d\omega \log(\omega^2) \omega^3 \rho(\omega) > 0.$$

Spectral moments

Postulate

Strong moment problem

$$\rho_0 \equiv \int d\omega \rho(w) = 1,$$

$$\rho_n \equiv \int d\omega \omega^n \rho(\omega) = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Observables are given by the inverse moments

Such a $\rho(\omega)$ exists!

$$\rho_{-k} \equiv \int d\omega \omega^{-k} \rho(\omega), \quad \text{for } k = 1, 2, 3, \ldots$$

as well as by the "log moments",

$$\rho'_n \equiv \int d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n = 2, 3, 4, \ldots$$

Gauge technique and the vertex functions

The vector and axial-vector currents of QCD are:

$$J_V^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \frac{\lambda_a}{2} q(x), \quad J_A^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x)$$

CVC and PCAC:

$$\partial_\mu J_V^{\mu,a}(x) = 0, \quad \partial_\mu J_A^{\mu,a}(x) = \bar{q}(x)\hat{M}_0 i\gamma_5 \frac{\lambda_a}{2} q(x)$$

These imply the Ward-Takahashi identities (WTI)

- Pions arise as Goldstone bosons, with standard current-algebra properties

for free!

- At high energies parton-model features, such as the spin-1/2 nature of hadronic constituents, are recovered

Gauge technique

Delburgo & West '77

The vector and axial unamputated vertex functions:

$$\Lambda^{\mu,a}_{V,A}(p',p) = \int d^4x d^4x' \langle 0|T\left\{J^{\mu,a}_{V,A}(0)q(x')\bar{q}(x)\right\}|0\rangle e^{ip'\cdot x' - ip\cdot x}$$

WTI

$$(p'-p)_\mu \Lambda^{\mu,a}_V(p',p) = S(p')\frac{\lambda_a}{2} - \frac{\lambda_a}{2}S(p)$$

$$(p'-p)_\mu \Lambda^{\mu,a}_A(p',p) = S(p')\frac{\lambda_a}{2}\gamma_5 + \gamma_5\frac{\lambda_a}{2}S(p)$$

The gauge technique consists of writing a solution

not unique!

$$\Lambda^{\mu,a}_V(p',p) = \int d\omega\rho(\omega)\frac{i}{\not p' - \omega}\gamma^\mu\frac{\lambda_a}{2}\frac{i}{\not p - \omega}$$

$$\Lambda^{\mu,a}_A(p',p) = \int d\omega\rho(\omega)\frac{i}{\not p' - \omega}\left(\gamma^\mu - \frac{2\omega q^\mu}{q^2}\right)\gamma_5\frac{\lambda_a}{2}\frac{i}{\not p - \omega}$$

pion pole

Vertices with two currents

Vertices with two currents, axial or vector, are constructed similarly. The vacuum polarization is

$$i\Pi^{\mu a,\nu b}_{VV}(q) = \delta^{ab}\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\bar{\Pi}_{VV}(q) = \int d^4x e^{-iq\cdot x}\langle 0|T\left\{J^{\mu a}_V(x)J^{\nu b}_V(0)\right\}|0\rangle$$
$$= -N_c\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not p-\not q-\omega}\gamma_\mu\frac{\lambda_a}{2}\frac{i}{\not p-\omega}\gamma_\nu\frac{\lambda_b}{2}\right]$$

transverse

$$\bar{\Pi}_{VV}(q) = \ldots$$
$$I(q^2,\omega) = -\frac{1}{(4\pi)^2}\int_0^1 dx\log\left[\omega^2+x(1-x)q^2\right]$$

Remark

The meaning of the gauge technique:

1) write the vertex 2) close the quark line

one $\rho(\omega)$ for each quark line

Dispersion relation

The twice-subtracted dispersion relation holds:

$$\bar{\Pi}_V(q^2) = \frac{q^4}{\pi} \int_0^\infty \frac{dt}{t^2} \frac{\mathrm{Im}\bar{\Pi}_V(t)}{t - q^2 - i0^+}$$

This is in contrast to quark models formulated in the Euclidean space, where the usual dispersion relations do not hold

$e^+e^- \to$ hadrons

At large s we find

proportional to $\mathrm{Im}\bar{\Pi}_V$

$$\sigma(e^+e^- \to \mathrm{hadrons}) = \frac{4\pi\alpha_{\mathrm{QED}}^2}{3s}\left(\sum_i e_i^2\right)\int d\omega\rho(\omega),$$

where e_i is the electric charge of the quark of species i. This is the proper asymptotic QCD result, provided

$$\int d\omega\rho(\omega) = 1$$

Pion weak decay

The pion decay constant, defined as

$$\langle 0 \,|J_A^{\mu a}(x)|\, \pi_b(q)\rangle = i f_\pi q_\mu \delta_{a,b} e^{iq\cdot x},$$

can be computed from the axial-axial correlation function.
The result is

$$f_\pi^2 \;=\; 4N_c \int d\omega \rho(\omega)\omega^2 I(0,\omega)$$

A finite value for f_π requires the condition $\rho_2 = 0$. Then

$$f_\pi^2 = -\frac{N_c}{4\pi^2}\int d\omega \log(\omega^2)\omega^2\rho(\omega) \equiv -\frac{N_c}{4\pi^2}\rho_2'$$

The value $f_\pi = 93\text{MeV}$ determines ρ_2'. The sign is

$$\rho_2' < 0$$

Pion electromagnetic form factor

The electromagnetic form factor for a positively charged pion, $\pi^+ = u\bar{d}$, is defined as

$$\langle \pi^+(p') | J_\mu^{\rm em}(0) | \pi^+(p) \rangle = (p^\mu + p'^\mu) e F_\pi^{\rm em}(q^2)$$

For on-shell massless pions the electromagnetic form factor reads

$$F_\pi^{em}(q^2) = \frac{4N_c}{f_\pi^2} \int dw \rho(\omega) \omega^2 I(q^2, \omega)$$

The low-momentum expansion is

$F_\pi^{em}(0) = 1$

$$F_\pi^{em}(q^2) = 1 + \frac{1}{4\pi^2 f_\pi^2} \left(\frac{q^2 \rho_0}{6} + \frac{q^4 \rho_{-2}}{60} + \frac{q^6 \rho_{-4}}{240} + \dots \right)$$

The mean squared radius reads

$$\langle r_\pi^2 \rangle = 6 \frac{dF}{dq^2}|_{q^2=0} = \frac{N_c}{4\pi^2 f_\pi^2} \int d\omega \rho(\omega) = \frac{N_c}{4\pi^2 f_\pi^2}$$

$$\langle r^2 \rangle_\pi^{\rm em} \Big|_{\rm th} = 0.34 {\rm fm}^2 \qquad \langle r^2 \rangle_\pi^{\rm em} \Big|_{\rm exp} = 0.44 {\rm fm}^2$$

which is reasonable (χPT corrections)

The knowledge of $F_\pi^{em}(q^2)$ allows us to determine the even negative moments of $\rho(\omega)$

Twist expansion and spectral conditions

In the limit of large momentum

$$F_\pi^{em}(q^2) \sim \frac{N_c}{4\pi^2 f_\pi^2} \int d\omega \rho(\omega) \{ \frac{2\omega^4}{q^2} \left[\log(-q^2/\omega^2) + 1 \right] + \ldots \}$$

With help of the spectral conditions for $n = 2, 4, 6, \ldots$ we get

All spectral conditions needed!

$$F_\pi^{em}(q^2) \sim -\frac{N_c}{4\pi^2 f_\pi^2} \left[\frac{2\rho_4'}{q^2} + \frac{2\rho_6'}{q^4} + \frac{4\rho_8'}{q^6} + \ldots \right]$$

The imposition of the spectral conditions removed all the logs from the expansion, leaving a pure expansion in inverse powers of q^2!

Anomalous form factor in $\pi^0 \to \gamma\gamma$

$$\Gamma^{\mu\nu}_{\pi^0\gamma\gamma}(q_1, q_2) = \epsilon_{\mu\nu\alpha\beta} q_1^\alpha p^\beta F_{\pi\gamma\gamma}(p, q_1, q_2) = -N_c \int d\omega \rho(w) \int \frac{d^4k}{(2\pi)^4} \times$$

$$\mathrm{Tr}\left[-\frac{\omega}{f_\pi}\gamma_5 \tau_0 \frac{i}{\not k - \not q_2 - \omega} i\hat{Q}\gamma^\mu \frac{i}{\not k - \omega} i\hat{Q}\gamma^\nu \frac{i}{\not k - \not q_1 - \omega}\right] + \mathrm{crossed}$$

where $\hat{Q} = N_c/2 + \tau_3/2$. We find

$$F_{\pi\gamma\gamma}(0,0,0) \quad = \quad \frac{1}{4\pi^2 f_\pi} \int d\omega \rho(\omega) = \frac{1}{4\pi^2 f_\pi}$$

which coincides with the standard QCD result!

Blin, Hiller & Schaden

Not true when the loop momentum is cut!

The $\gamma \to \pi^+\pi^0\pi^-$ decay

Consider an example of a low-energy process involving a quark box diagram which is related to the QCD anomaly. The amplitude for $\gamma(q,e) \to \pi^+(p_1)\pi^0(p_2)\pi^-(p_3)$ is

$$T_{\gamma(q,\varepsilon)\to\pi^+(p_1)\pi^0(p_2)\pi^-(p_3)} \equiv F(p_1,p_2,p_3)\varepsilon_{\alpha\beta\sigma\tau}e^\alpha p_1^\beta p_2^\sigma p_3^\tau.$$

In the limit of all momenta going to zero we get, with the spectral normalization condition,

$$F(0,0,0) = \frac{1}{4\pi^2 f_\pi^3}\int d\omega\rho(\omega) = \frac{1}{4\pi^2 f_\pi^3}$$

which is the correct result.

Pion-photon transition form factor

For two off-shell photons with momenta q_1 and q_2 one defines the asymmetry, A, and the total virtuality, Q^2:

$$\begin{aligned} A &= \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \qquad -1 \leq A \leq 1 \\ Q^2 &= -(q_1^2 + q_2^2) \end{aligned}$$

At the soft pion point we find the expansion,

$$F_{\pi\gamma\gamma}(Q^2, A) = -\frac{1}{2\pi^2 f_\pi} \int_0^1 dx \left[\frac{2\rho_2'}{Q^2(1 - A^2(2x-1)^2)} + \dots \right]$$

We can confront this with the standard twist decomposition of the pion transition form factor ,

$$F_{\gamma\gamma\pi}(Q^2, A) = J^{(2)}(A)\frac{1}{Q^2} + J^{(4)}(A)\frac{1}{Q^4} + \dots,$$

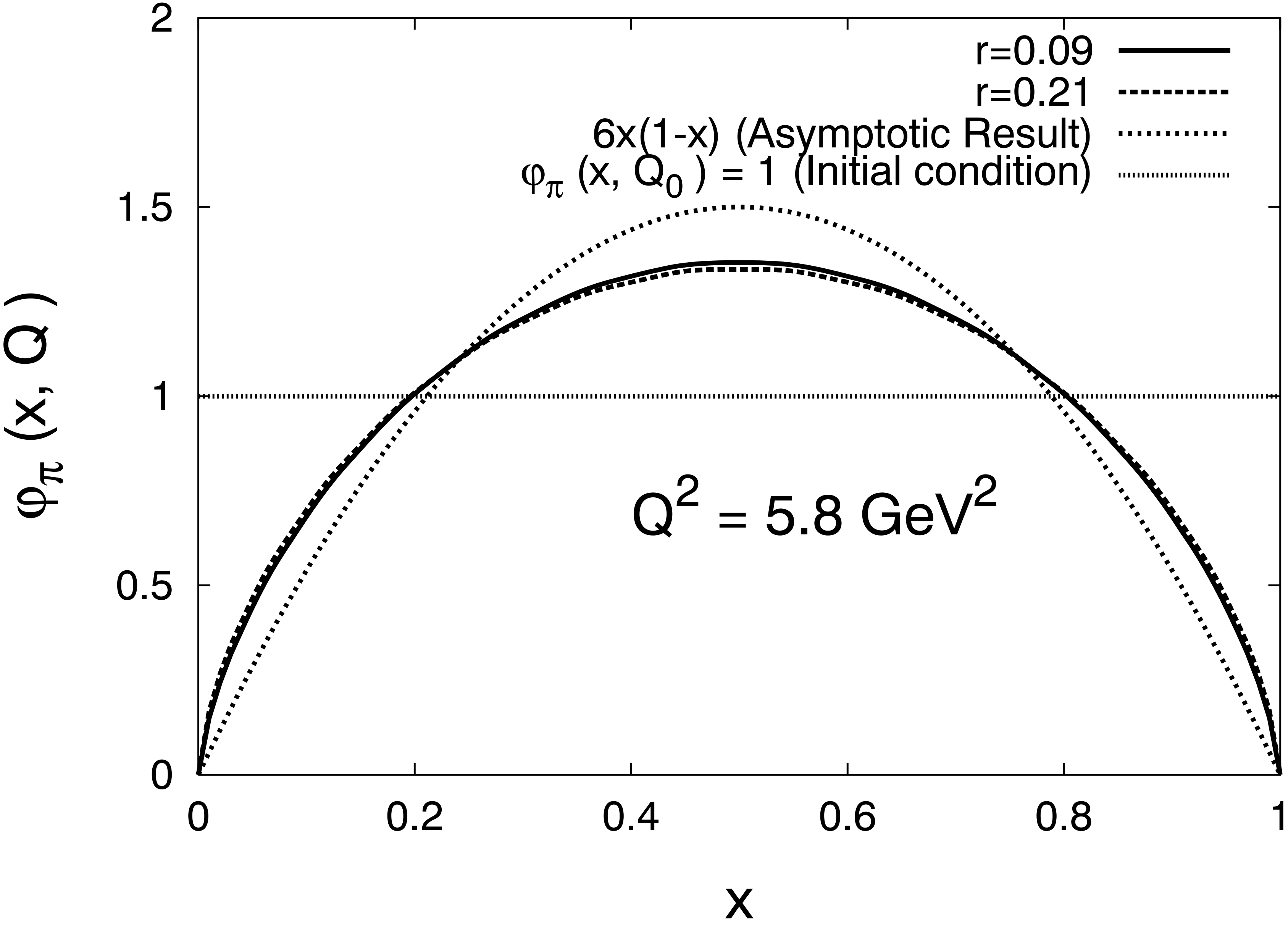
Brodsky-Lepage, Praszałowicz-Rostworowski, Dorokhov which yields

$$J^{(2)}(A) = \frac{4f_\pi}{N_c}\int_0^1 dx \frac{\varphi(x;Q_0)}{1-(2x-1)^2A^2}$$

with the leading-twist pion distribution amplitude (PDA) given by

$$\varphi(x;Q_0) = 1$$

This serves as the initial conditiuon for the QCD evolution



r=0.09
r=0.21
6x(1-x) (Asymptotic Result)
φ_π (x, Q_0) = 1 (Initial condition)
$Q^2 = 5.8\ \mathrm{GeV}^2$
φ_π (x, Q)
x
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

Pion light-cone wave function

$$\Psi(x,k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}\int d\omega \rho(\omega)\frac{\omega^2}{k_\perp^2+\omega^2}\theta(x)\theta(1-x)$$

(at the low-energy scale of the model, Q_0). It has correct normalization, since $\int d^2k_\perp \Psi(x,k_\perp) = \varphi(x) = 1$. At $k_\perp = 0$ it satisfies the condition:

$$\Psi(x,0) = \frac{N_c}{\pi f_\pi}F_{\pi\gamma\gamma}(0,0,0) = \frac{N_c}{4\pi^3 f_\pi^2}$$

In QCD a similar relation holds for quantities integrated over x.

Pion structure function

Davidson & Arriola in NJL

We take π^+ for definiteness and get

$$u_\pi(x) = \bar{d}_\pi(1-x) = \theta(x)\theta(1-x),$$

independently of the spectral function $\rho(\omega)$. One recovers scaling in the Bjorken limit, the Callan-Gross relation, the proper support, and the correct normalization

The $k_\perp$-unintegrated parton distribution can be shown to be equal to

$$q(x,k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2} \int d\omega \rho(\omega) \frac{\omega^2}{k_\perp^2 + \omega^2} \theta(x)\theta(1-x),$$

hence at Q_0 one has an interesting relation

$$q(x,k_\perp) = \bar{q}(1-x,k_\perp) = \Psi(x,k_\perp).$$

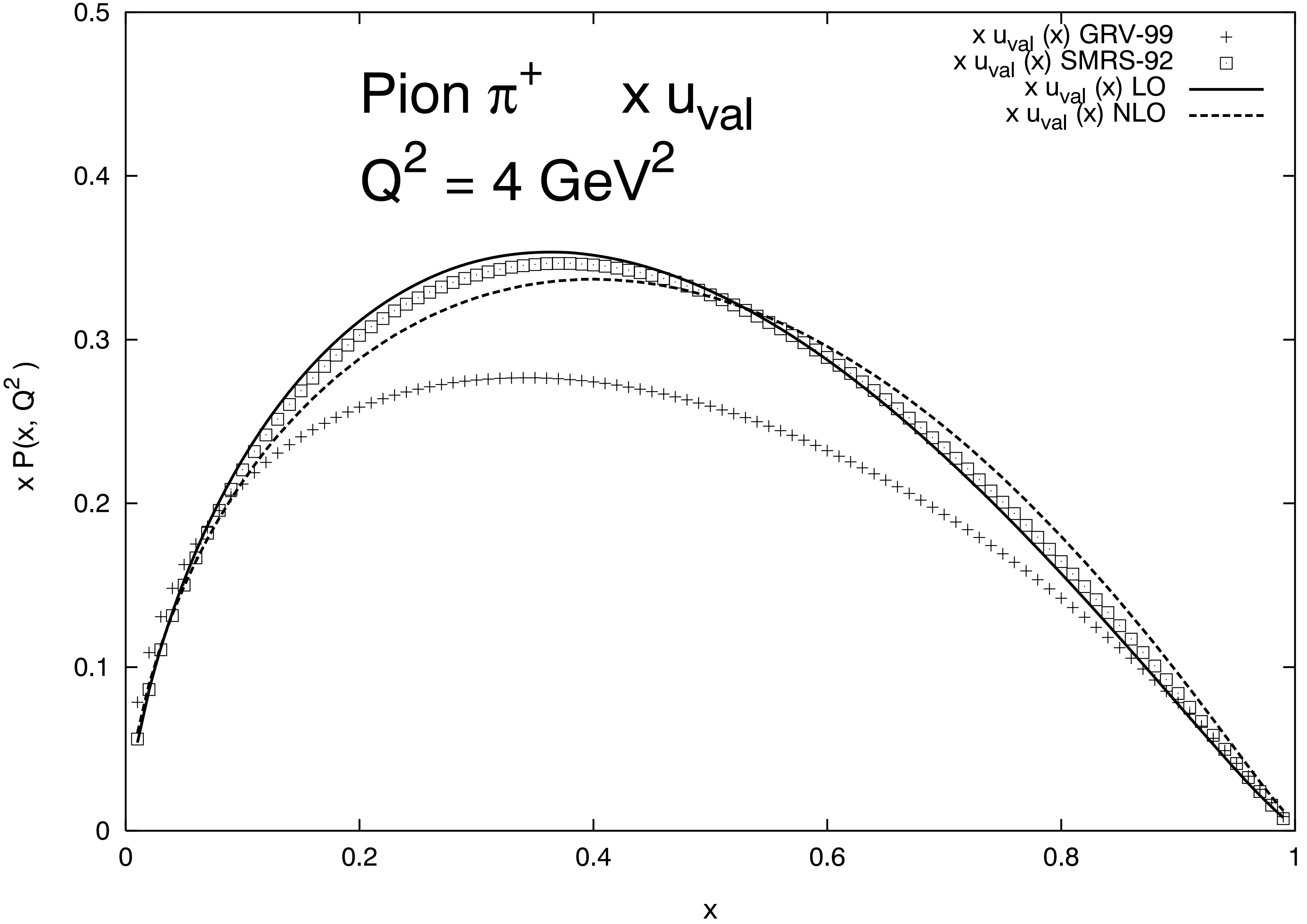
At $k_\perp = 0$ we have

$$q(x, 0_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}.$$

The first moment of the PDF is responsible for the momentum sum rule. We find

quarks carry all momentum

$$\int_0^1 dx\, xq(x) = \int_0^1 dx\, x\bar{q}(x) = \frac{1}{2}$$


Pion π+ x uval
Q2 = 4 GeV2
x uval (x) GRV-99
x uval (x) SMRS-92
x uval (x) LO
x uval (x) NLO
x P(x, Q2)
x
0
0.1
0.2
0.3
0.4
0.5
0
0.2
0.4
0.6
0.8
1

Further results/predictions

Gasser-Leutwyler coefficients:

$$\bar{l}_1 = -N_c, \quad \bar{l}_2 = N_c, \quad \bar{l}_5 - 1 = 2N_c, \quad \bar{l}_6 - 1 = 4N_c$$

unregularized quark-model values

Charged pion polarizability:

$$\alpha_{\pi^\pm} = \frac{\alpha_{QED}(\bar{l}_6 - \bar{l}_5)}{48\pi^2 m_\pi f_\pi^2} = 5.9 \times 10^{-3} \mathrm{fm}^3$$

Magnetic permeability of the vacuum, χ

$$\langle 0|\bar{q}(0)\sigma_{\alpha\beta}q(0)|\gamma^{(\lambda)}(q)\rangle = ie_q\chi\,\langle\bar{q}q\rangle\left(q_\beta\varepsilon_\alpha^{(\lambda)} - q_\alpha\varepsilon_\beta^{(\lambda)}\right)$$

We find

First log-moment

$$\chi = \frac{N_c}{4\pi^2}\rho_1'/\langle\bar{q}q\rangle$$

Tensor susceptibility of the vacuum

$$\Pi = i\langle 0|\int d^4z\, T\{\overline{q}(z)\sigma^{\mu\nu}q(z), \overline{q}(0)\sigma_{\mu\nu}q(0)\}|0\rangle$$

The model yields

Broniowski, Polyakov & Goeke '98

$$\Pi = -12 f_\pi^2$$

Résumé

Spectral condition	Physical significance
normalization	
$\rho_0 = 1$	proper normalization of the quark propagator
	preservation of anomalies
	proper normalization of the pion distribution amplitude
	proper normalization of the pion structure function
	reproduction of the large-N_c quark-model values of the Gasser-Leutwyler coefficients
	relation $M_V^2 = 24\pi^2 f_\pi^2 / N_c$ in the VMD model
positive moments	
$\rho_1 = 0$	finiteness of the quark condensate, $\langle \bar{q} q \rangle$
	vanishing quark mass at asymptotic Euclidean momenta,
$\rho_2 = 0$	finiteness of the vacuum energy density, B
	finiteness of the pion decay constant, f_π
$\rho_3 = 0$	finiteness of the quark condensate, $\langle \bar{q} q \rangle$
$\rho_4 = 0$	finiteness of the vacuum energy density, B
$\rho_n = 0, \; n = 2, 4 \ldots$	absence of logs in the twist expansion of vector amplitudes
$\rho_n = 0, \; n = 5, 7 \ldots$	finiteness of nonlocal quark condensates, $\langle \bar{q} (\partial^2)^{(n-3)/2} q \rangle$
	absence of logs the twist expansion of the scalar pion form factor

Spectral condition	Physical significance
negative moments	
$\rho_{-2} > 0$	positive quark wave-function normalization at vanishing momentum
$\rho_{-1}/\rho_{-2} > 0$	positive value of the quark mass at vanishing momentum, $M(0) > 0$
ρ_{-n}	low-momentum expansion of correlators
log-moments	
$\rho'_2 < 0$	$f_\pi^2 = -N_c/(4\pi^2)\rho'_2$
$\rho'_3 > 0$	negative value of the quark condensate, $\langle \bar{q}q \rangle = -N_c/(4\pi^2)\rho'_3$
$\rho'_4 > 0$	negative value of the vacuum energy density, $B = -N_c/(4\pi^2)\rho'_4$
$\rho'_5 < 0$	positive value of the squared vacuum virtuality of the quark, $\lambda_q^2 = -\rho'_5/\rho'_3$
ρ'_n	high-momentum (twist) expansion of correlators

Meson dominance model

Now we construct explicitly an example of $\rho(\omega)$. Vector-meson dominance (VMD) of the pion form factor is assumed,

$$F_V(t) = \frac{M_V^2}{M_V^2 + t}$$

with $M_V = m_\rho$. In the spectral approach

$$F_V(t) = \frac{N_c}{4\pi^2 f_\pi^2} \sum_{n=1}^{\infty} \rho_{2-2n} \frac{2^{-2n-1}\sqrt{\pi}\Gamma(n+1)}{n\Gamma(n+3/2)} (-t)^n$$

Comparison yields

$$\rho_{2-2k} = \frac{2^{2k+3}\pi^{3/2} f_\pi^2}{N_c M_V^{2k}} \frac{k\,\Gamma(k+3/2)}{\Gamma(k+1)}, k = 1, 2, 3, \ldots$$

The normalization condition, $\rho_0 = 1$, gives

$$M_V^2 = \frac{24\pi^2 f_\pi^2}{N_c}$$

This relation is usually obtained when matching chiral quark models to VMD **Scadron!**

Even though we have determined the negative even moments of the spectral function, the positive even moments fulfil the spectral conditions for the positive moments since $\Gamma(n)$ has poles at non-positive integers, $n = 0, -1, -2, \ldots$ **Miracle!**

$$\rho_{2n} = 0, \qquad n = 1, 2, 3 \ldots$$

For the log-moments we have

$$\rho'_{2n} = \left(-\frac{M_V^2}{4}\right)^n \frac{\Gamma(n)\,\Gamma\left(\frac{5}{2}-n\right)}{\Gamma(\frac{5}{2})}, \qquad n = 1, 2, 3 \ldots$$

Inverse problem

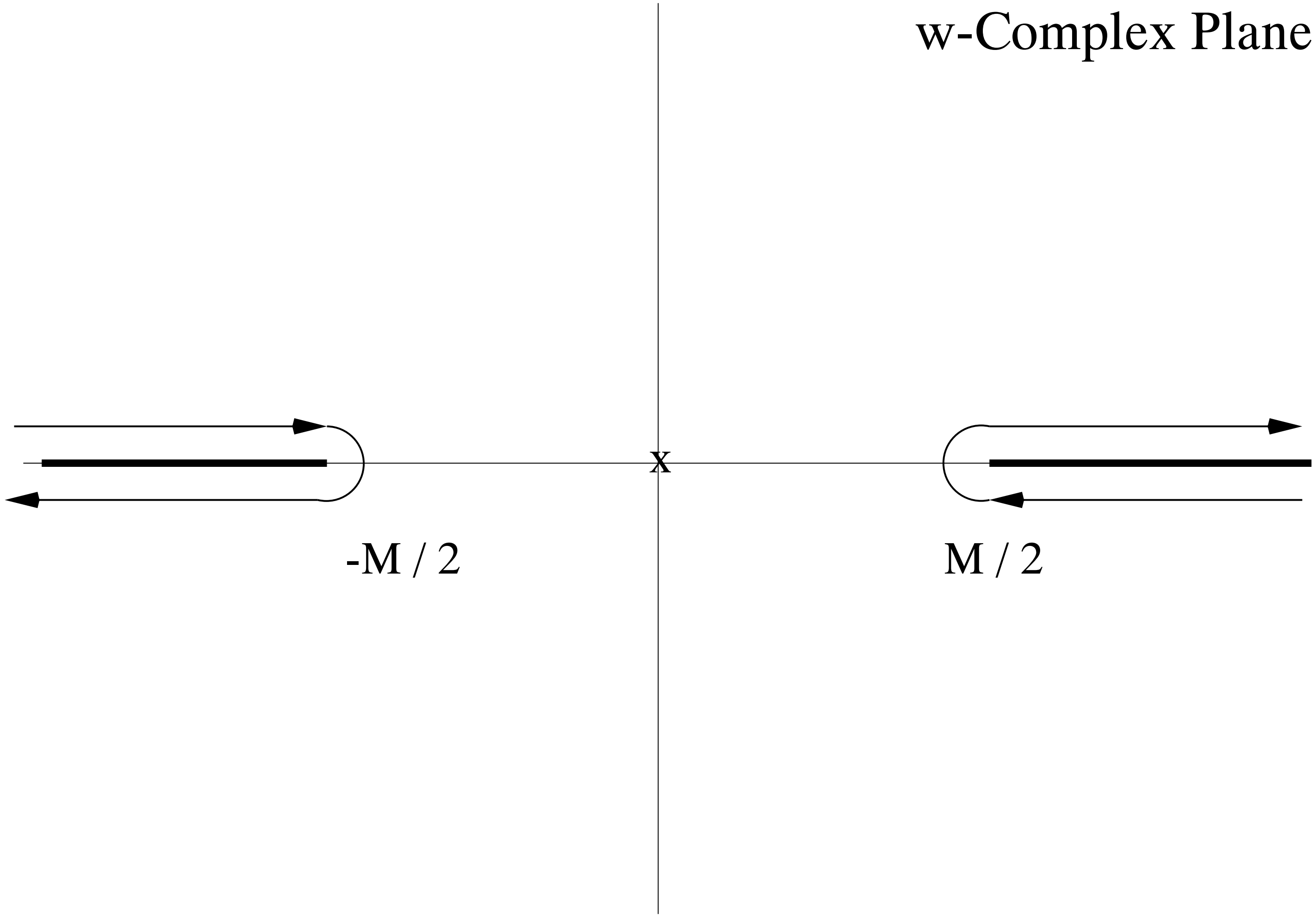
The mathematical problem is now to invert the formula

$$\rho_{2n} = \int_C d\omega \omega^{2n} \rho_V(\omega)$$

The solution is given by the following surprisingly simple function

$$\rho_V(\omega) = \frac{1}{2\pi i} \frac{3\pi^2 M_V^3 f_\pi^2}{4N_c} \frac{1}{\omega} \frac{1}{(M_V^2/4 - \omega^2)^{5/2}}.$$

The function $\rho_V(\omega)$ has a single pole at the origin and branch cuts starting at $\pm$ half the meson mass, $\omega = \pm M_V/2$.



contour C

Scalar spectral function

For the case of the scalar spectral function (controlling odd moments) we proceed heuristically, by proposing its form in analogy to ρ_V

$\rho = \rho_V + \rho_S$

$$\rho_S(\omega) = \frac{1}{2\pi i} \frac{16(d_S-1)(d_S-2)\rho_3'}{M_S^4(1-4\omega^2/M_S^2)^{d_S}}$$

Normalization is chosen such that $\rho_3' = -4\pi^2 \langle \bar{q} q \rangle / N_c$. The admissible d_S is half-integer, since only then the integration around the half-circles at the branch points vanishes. The preferred value is $d_S = 5/2$. The analytic structure similar to $\rho_V(\omega)$, except for the absence of the pole at $\omega = 0$.

Quark propagator

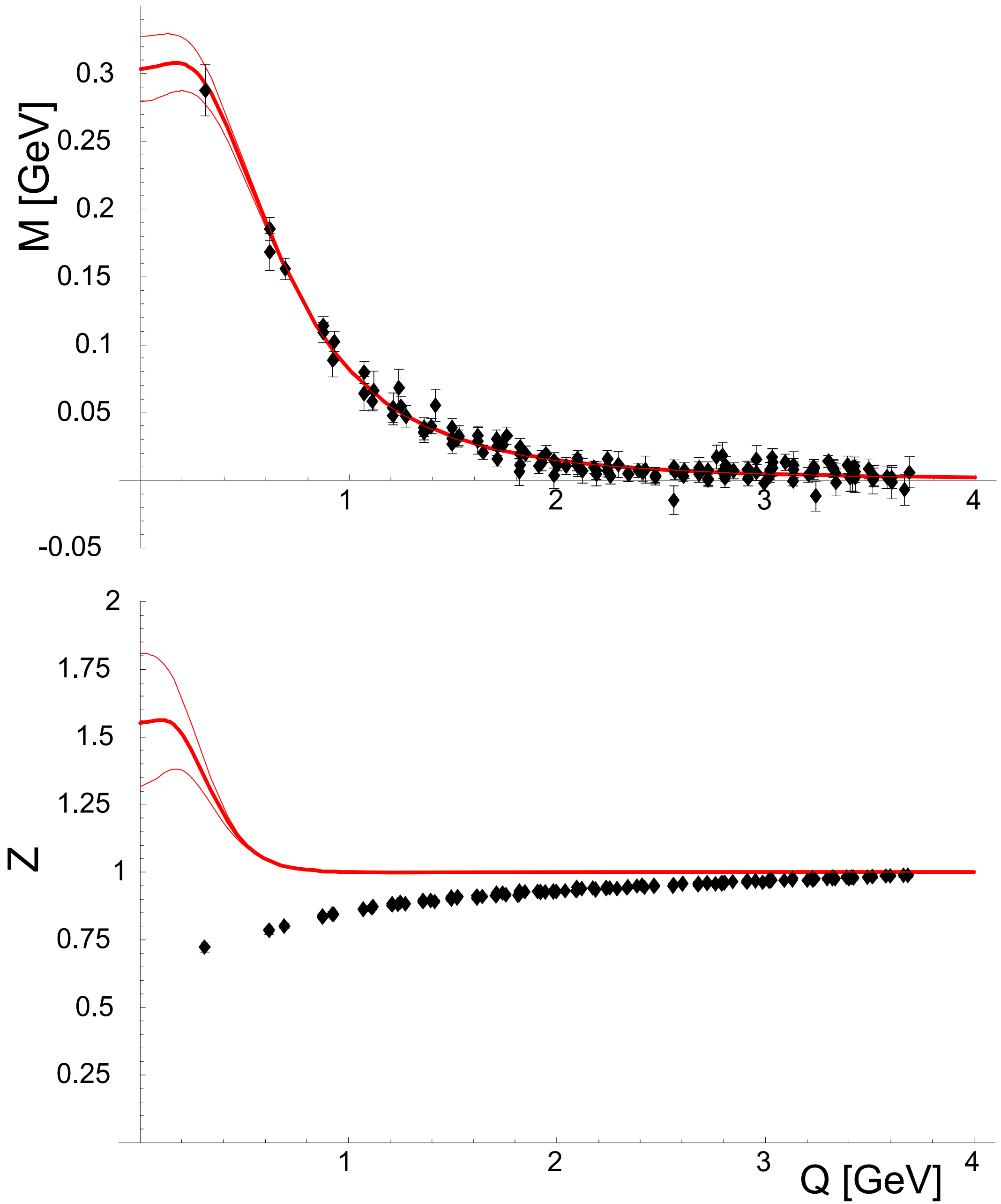
$$S(p) = A(p)\not{p} + B(p) = Z(p)\frac{\not{p} + M(p)}{p^2 - M^2(p)}$$

$$A(p^2) \equiv \int_C d\omega \frac{\rho_V(\omega)}{p^2 - \omega^2} = \frac{1}{p^2}\left[1 - \frac{1}{(1 - 4p^2/M_V^2)^{d_V}}\right]$$

$$B(p^2) \equiv \int_C d\omega \frac{\omega \rho_S(\omega)}{p^2 - \omega^2} = \frac{64(d_S - 2)(d_S - 1)\pi^2 \langle \bar{q}q \rangle}{M_S^4 N_c (1 - 4p^2/M_S^2)^{d_S}}$$

No poles in the whole complex plane! Only branch cuts starting at $p^2 = 4M^2$, where M is the relevant mass. The absence of poles is sometimes called "analytic confinement"

Poles would lead to cuts in form f.



data:

Bowman, Heller,

& Williams '02

$M(Q^2)$ decreases as $1/Q^3$ at large Euclidean momenta, which is favored by the recent lattice calculations. The χ^2 fit results in the following optimum values,

$$
\begin{aligned}
M_0 &= 303 \pm 24 \text{ MeV}, \\
M_S &= 970 \pm 21 \text{ MeV},
\end{aligned}
$$

with $\chi^2/\text{DOF} = 0.72$. The corresponding value of the quark condensate is

$$
\langle \bar{q} q \rangle = -(243.0^{+0.1}_{-0.8} \text{ MeV})^3
$$

Other predictions

Pion light-cone wave function and PDF:

$$\Psi(x,k_\perp)=q(x,k_\perp)=\frac{3M_V^3}{16\pi(k_\perp^2+M_V^2/4)^{5/2}}\theta(x)\theta(1-x)$$

Passing to the transverse-coordinate representation

$$\Psi(x,b)=\left(1+\frac{bM_V}{2}\right)\exp\left(-\frac{M_V b}{2}\right)\theta(x)\theta(1-x)$$

Quark propagator in the coordinate representation:

$$A(z) = \frac{48+24M_V\sqrt{-z^2}-6M_V^2z^2+M_V^3(-z^2)^{3/2}}{96\pi^2z^4}\exp(-M_V\sqrt{-z^2}/2)$$

$$B(z) = \langle\overline{q}q\rangle/(4N_c)\exp(-M_S\sqrt{-z^2}/2)$$

Nonlocal quark condensate:

$$Q(z) = \exp(-M_S\sqrt{-z^2}/2)$$

Magnetic permeability of the vacuum (at Q_0):

$$\chi = \frac{2}{M_S^2}$$

With $M_S = 0.97$ GeV this yields, after evolution,

c.f. Ball, Braun & Kivel '03

$$\chi(1 \text{ GeV}) = 3.3 \text{ GeV}^2$$

in agreement with other estimates

Final remarks

1. What has been done? Spectral representation, one loop (large N_c), gauge technique (WTI), spectral conditions (finiteness)

2. Symmetries, anomalies, normalization, pure twist expansion, guaranteed

3. Dynamics encoded in moments, the approach itself is non-dynamical

4. The method is very simple (computations are short) and predictive (lots of applications, many relations follow naturally)

5. What does not work (at the moment): Weinberg sum rule II (modify vertices)

6. General remark: need for the QCD evolution of PDA, PDF, ..., in order to pass from the model's scale (low) to the high scale of DIS data. Good results follow

7. Interesting particular realization of the spectra idea: the meson-dominance model

8. Analytic confinement in the sense of the absence of poles in the quark propagator

9. Surprisingly good $M(Q^2)$ vs. lattice results, $Z(Q^2)$ could be better

BACKUP SLIDES

Vacuum energy density

$$\langle \theta^{\mu\nu} \rangle = -iN_cN_f \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \times$$
$$\mathrm{Tr} \frac{1}{\not{p} - \omega} \left[\frac{1}{2} (\gamma^\mu p^\nu + \gamma^\nu p^\mu) - g^{\mu\nu} (\not{p} - \omega) \right] = B g^{\mu\nu} + \langle \theta^{\mu\nu} \rangle_0,$$

where $\langle \theta^{\mu\nu} \rangle_0$ is the energy-momentum tensor for the free theory, evaluated with $\rho(\omega) = \delta(\omega)$, and B (bag constant) is the vacuum energy density:

$$B = -iN_cN_f \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega^2}{p^2 - \omega^2},$$

The conditions that have to be fulfilled for B to be finite are

$$\rho_2 = 0, \quad \rho_4 = 0$$

Then

$$B = -\frac{N_c N_f}{16\pi^2}\rho_4' \equiv -\frac{3N_c}{16\pi^2}\int d\omega \log(\omega^2)\omega^4\rho(\omega)$$

for $N_f = 3$.

According to QCD sum rules

$$B = -\frac{9}{32}\langle\frac{\alpha}{\pi}G^2\rangle = -(224^{+35}_{-70}\mathrm{MeV})^4$$

The negative sign of B enforces

$$\rho_4' > 0$$

Pion-quark coupling

Near the pion pole ($q^2 = 0$) we get

$$\Lambda_A^{\mu,a}(p+q,p) \to -\frac{q^\mu}{q^2}\Lambda_\pi^a(p+q,p),$$

where

$$\Lambda_\pi^a(p+q,p) = \int d\omega\rho(\omega)\frac{i}{\not p + \not q - \omega}\frac{\omega}{f_\pi}\gamma_5\lambda_a\frac{i}{\not p - \omega}$$

We recognize in our formulation the Goldberger-Treiman relation for quarks:

$$g_\pi(\omega) = \frac{\omega}{f_\pi}$$

QCD evolution of PDA

All results of the effective, low-energy model, refer to a soft energy scale, Q_0. In order to compare to experimental results, obtained at large scales, Q, the QCD evolution must be performed. Initial condition:

$$\varphi(x;Q_0) = \theta(x)\theta(1-x).$$

The evolved distribution amplitude reads

$$\begin{aligned} \varphi(x;Q) &= 6x(1-x)\sum_{n=0}^{\infty} C_n^{3/2}(2x-1)a_n(Q) \\ a_n(Q) &= \frac{2}{3}\frac{2n+3}{(n+1)(n+2)}\left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)}\right)^{\gamma_n^{(0)}/(2\beta_0)} \end{aligned}$$

where $C_n^{3/2}$ are the Gegenbauer polynomials, $\gamma_n^{(0)}$ are appropriate anomalous dimensions, and $\beta_0 = 9$.

Results extracted from the experimental data of CLEO provide $a_2(2.4\mathrm{GeV}) = 0.12 \pm 0.03$, which we use to fix

$$\alpha(Q = 2.4\mathrm{GeV})/\alpha(Q_0) = 0.15 \pm 0.06$$

At LO this corresponds to $Q_0 = 322 \pm 45$ MeV

Now we can predict

$$\begin{aligned} a_4(2.4\mathrm{GeV}) &= 0.06 \pm 0.02 \;\; (\mathrm{exp}: -0.14 \pm 0.03 \mp 0.09) \\ a_6(2.4\mathrm{GeV}) &= 0.02 \pm 0.01 \end{aligned}$$

Encouraging, with leading-twist and LO QCD evolution!

QCD evolution of PDF

The QCD evolution of the constant PDF has been treated in detail by Davidson & ERA at LO and NLO. In particular, the non-singlet contribution to the energy-momentum tensor evolves as

$$\frac{\int dx\, xq(x,Q)}{\int dx\, xq(x,Q_0)} = \left(\frac{\alpha(Q)}{\alpha(Q_0)}\right)^{\gamma_1^{(0)}/(2\beta_0)} ,$$

In has been found that at $Q^2 = 4\mathrm{GeV}^2$ the valence quarks carry $47 \pm 0.02\%$ of the total momentum fraction in the pion. Downward LO evolution yields that at the scale

$$Q_0 = 313^{+20}_{-10}\mathrm{MeV}$$

the quarks carry 100% of the momentum. The agreement of the evolved PDF with the SMRS data analysis is impressive

Gasser-Leutwyler coefficients

The one-quark-loop effective action that incorporates the quark-pion coupling obeying the Goldberger-Treiman relation can be written as

$$S = -iN_c \int d^4x \int d\omega \rho(\omega) \mathrm{Tr} \log \left[i\partial\!\!\!/ - \omega \exp\left(i\gamma_5 \tau_a \phi_a(x)/f_\pi \right) \right]$$

This form is manifestly chirally symmetric, with ϕ denoting the non-linearly realized pion field. One may evaluate the Gasser-Leutwyler coefficients through the use of standard derivative expansion techniques. With the spectral normalization condition imposed, the calculation is equivalent to standard quark-model calculations with the cut-off removed. One gets

$$\bar{l}_1 \;=\; -N_c,$$

$$\bar{l}_2 = N_c.$$

Other low energy constants, such as $\bar{l}_3$ and $\bar{l}_4$ require a specification of explicit chiral symmetry breaking within the quark model.