Derivation of transport equation

Ludwig Boltzmann (1844-1906) was presumably the first who formulated a program to derive principles of statistical mechanics from laws of microscopic underlying dynamics. Nikolay Nikolayevich Bogolyubov (1909-1992) succeeded to derive the equations of kinetic theory from the equations of classical Newtonian mechanics. Later on, an analogous derivation was presented starting with quantum mechanics. Since quantum field theory is currently the most fundamental formulation of microscopic dynamics, the laws of statistical physics should be derived from it.

In this lecture the kinetic equation is derived from the exact equations of motion of quantum field theory using the real-time formalism of statistical QFT. We consider again a system described by the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x), \tag{1}$$

where $\phi(x)$ is the real scalar, m is the mass parameter and λ is the coupling constant.

Exact equations of motion

- The starting point of the derivation are the exact equations of motion of contour Green's function which read

$$\big(\Box_x + m^2\big)\Delta(x,y) = -\delta_C^{(4)}(x,y) + \int_C d^4x'\,\Pi(x,x')\,\Delta(x',y), \tag{2}$$

$$\big(\Box_y + m^2\big)\Delta(x,y) = -\delta_C^{(4)}(x,y) + \int_C d^4x'\,\Delta(x,x')\,\Pi(x',y), \tag{3}$$

  where the contour Dirac delta is defined as

$$\delta_C^{(4)}(x,y) = \begin{cases} \delta^{(4)}(x-y) & \text{for} \quad x_0, y_0 \quad \text{from the upper branch,} \\ 0 & \text{for} \quad x_0, y_0 \quad \text{from the different branches,} \\ -\delta^{(4)}(x-y) & \text{for} \quad x_0, y_0 \quad \text{from the lower branch,} \end{cases} \tag{4}$$

  and $\Pi(x,y)$ is the self energy.

- The kinetic equation will be obtained from the equations of motion of the real-time functions $\Delta^{\gtrless}$. One gets the equations from Eqs. (2, 3) using the decomposition

$$\Pi(x,y) = \delta_C^{(4)}(x,y)\Pi_\delta(x) + \Theta_C(x_0,y_0)\,\Pi^>(x,y) + \Theta_C(y_0,x_0)\,\Pi^<(x,y), \tag{5}$$

  where the one-point contribution $\Pi_\delta(x)$, which is generated by tadpole diagrams, is singled out. One also observes that the contour self energy equals

$$\Pi(x,y) = \delta_C^{(4)}(x,y)\Pi_\delta(x) + \begin{cases} \Pi^>(x,y) & \text{for} \quad x_0 \in C^- \quad \& \quad y_0 \in C^+, \\ \Pi^<(x,y) & \text{for} \quad x_0 \in C^+ \quad \& \quad y_0 \in C^-, \\ \Pi^c(x,y) & \text{for} \quad x_0 \in C^+ \quad \& \quad y_0 \in C^+, \\ \Pi^a(x,y) & \text{for} \quad x_0 \in C^- \quad \& \quad y_0 \in C^-, \end{cases} \tag{6}$$

  where C^+ and C^- denote the upper and the lower branch of the Keldysh contour.

- Putting x_0 on the lower branch of the contour and y_0 on the upper branch, we obtain the equations of $\Delta^>(x,y)$

$$\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^>(x,y) = \int d^4x' \Big(\Pi^>(x,x')\,\Delta^c(x',y) - \Pi^a(x,x')\,\Delta^>(x',y)\Big), \tag{7}$$

$$\big(\Box_y + m^2 - \Pi_\delta(y)\big)\Delta^>(x,y) = \int d^4x' \Big(\Delta^>(x,x')\,\Pi^c(x',y) - \Delta^a(x,x')\,\Pi^>(x',y)\Big), \tag{8}$$

where the decomposition (5) has been used.

- The equations of motion of $\Delta^<(x,y)$ are

$$\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^<(x,y) = \int d^4x' \Big(\Pi^c(x,x')\,\Delta^<(x',y) - \Pi^<(x,x')\,\Delta^a(x',y)\Big), \tag{9}$$

$$\big(\Box_y + m^2 - \Pi_\delta(y)\big)\Delta^<(x,y) = \int d^4x' \Big(\Delta^c(x,x')\,\Pi^<(x',y) - \Delta^<(x,x')\,\Pi^a(x',y)\Big). \tag{10}$$

- The equations (7 - 10) will be rewritten using the retarded and advanced Green's functions defined as

$$\Delta^+(x,y) \equiv \Theta(x_0 - y_0)\big(\Delta^>(x,y) - \Delta^<(x,y)\big), \tag{11}$$

$$\Delta^-(x,y) \equiv -\Theta(y_0 - x_0)\big(\Delta^>(x,y) - \Delta^<(x,y)\big). \tag{12}$$

- Keeping in mind the definitions of the chronological (Feynman) and antichronological functions

$$\Delta^c(x,y) \equiv \Theta(x_0 - y_0)\,\Delta^>(x,y) + \Theta(y_0 - x_0)\,\Delta^<(x,y), \tag{13}$$

$$\Delta^a(x,y) \equiv \Theta(y_0 - x_0)\,\Delta^>(x,y) + \Theta(x_0 - y_0)\,\Delta^<(x,y), \tag{14}$$

one finds the following identities

$$\Delta^c(x,y) = \Delta^+(x,y) + \Delta^<(x,y), \qquad \Delta^a(x,y) = -\Delta^+(x,y) + \Delta^>(x,y), \tag{15}$$

$$\Delta^c(x,y) = \Delta^-(x,y) + \Delta^>(x,y), \qquad \Delta^a(x,y) = -\Delta^-(x,y) + \Delta^<(x,y). \tag{16}$$

- Using the identities (15, 16) and the analogous relations among the self energies, one rewrites the equations (7 - 10) as

$$\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^{\gtrless}(x,y) = \int d^4x' \Big(\Pi^{\gtrless}(x,x')\,\Delta^-(x',y) + \Pi^+(x,x')\,\Delta^{\gtrless}(x',y)\Big), \tag{17}$$

$$\big(\Box_y + m^2 - \Pi_\delta(y)\big)\Delta^{\gtrless}(x,y) = \int d^4x' \Big(\Delta^{\gtrless}(x,x')\,\Pi^-(x',y) + \Delta^+(x,x')\,\Pi^{\gtrless}(x',y)\Big). \tag{18}$$

The kinetic equation will be obtained from the equations (17, 18).

Wigner transformation

- Since a non-equilibrium system to be described by the kinetic equation is, in general, inhomogeneous, the translational invariance is broken. So, we introduce the variables

$$X \equiv \frac{1}{2}(x+y), \qquad u \equiv x - y, \tag{19}$$

and the Green's functions are written as

$$\Delta(x,y) = \Delta\big(X + \frac{1}{2}u, X - \frac{1}{2}u\big) \equiv \Delta(X,u). \tag{20}$$

- Instead of the Fourier transformation we will use the Wigner transformation which is defined as

$$\Delta(X,p) = \int d^4u\, e^{ip\cdot u} \Delta\Big(X + \frac{1}{2}\,u, X - \frac{1}{2}\,u\Big). \tag{21}$$

The inverse transformation is

$$\Delta\Big(X + \frac{1}{2}\,u, X - \frac{1}{2}\,u\Big) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot u} \Delta(X,p). \tag{22}$$

- The first step in converting the equations of motion (17, 18) into the transport equation is to perform the Wigner transformation. We take this step assuming that the system is weakly inhomogeneous.

Weak inhomogeneity

- The system is assumed to be weakly inhomogeneous in comparison with two different length scales.

1. The inhomogeneity length is large compared to the inverse of the characteristic momentum *i.e.*

$$|F(X,p)| \gg \Big|\frac{\partial^2}{\partial X_\mu \partial p_\nu} F(X,p)\Big|, \tag{23}$$

where F is either the Green's function or self-energy. This assumption allows one to perform the gradient expansion.

2. The inhomogeneity length is assumed to be large compared to the Comptom wavelength (the inverse characteristic mass) *i.e.*

$$|F(X,p)| \gg \frac{1}{m^2}\Big|\frac{\partial^2}{\partial X_\mu \partial X^\mu} F(X,p)\Big|. \tag{24}$$

This condition justifies the quasiparticle approximation. When the bare fields are massless or the free mass is much smaller than the dynamically generated effective mass m_*, the mass m should be replaced by m_*.

Approximations

- The equations (17 - 18) are converted into the transport equation by performing Wigner transformation (21) of all Green's functions and self-energies. Using the condition (23), we obtain the following set of approximate translation rules:

$$
\begin{aligned}
\int d^4x' f(x,x')g(x',y) &\longrightarrow f(X,p)g(X,p) + \frac{i}{2}\{f(X,p),\, g(X,p)\}, \qquad (25)\\
h(x)g(x,y) &\longrightarrow h(X)g(X,p) - \frac{i}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu},\\
h(y)g(x,y) &\longrightarrow h(X)g(X,p) + \frac{i}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu},\\
\partial_x^\mu f(x,y) &\longrightarrow (-ip^\mu + \frac{1}{2}\partial^\mu)f(X,p),\\
\partial_y^\mu f(x,y) &\longrightarrow (ip^\mu + \frac{1}{2}\partial^\mu)f(X,p),
\end{aligned}
$$

where we have introduced the Poisson-like bracket defined as

$$
\Big\{C(X,p),\, D(X,p)\Big\} \equiv \frac{\partial C(X,p)}{\partial p_\mu}\frac{\partial D(X,p)}{\partial X^\mu} - \frac{\partial C(X,p)}{\partial X^\mu}\frac{\partial D(X,p)}{\partial p_\mu}\ .
$$

The function $h(x)$ is weakly dependent on x and we use the notation $\partial^\mu \equiv \frac{\partial}{\partial X_\mu}$.

- The condition (24) allows one to drop the terms containing ∂^2.

Transport and mass-shell equation

- Applying the translation rules (25) to Eqs. (17, 18), neglecting the terms proportional to ∂^2 due to the quasi-particle approximation (24), and taking the difference and sum of the equations, we obtain the transport and mass-shell equations

$$
\begin{aligned}
\Big[p^\mu\partial_\mu - \frac{1}{2}\partial_\mu\Pi_\delta(X)\partial_p^\mu\Big]\Delta^{\gtrless}(X,p) &= \frac{i}{2}\Big(\Pi^>(X,p)\Delta^<(X,p) - \Pi^<(X,p)\Delta^>(X,p)\Big)\\
&- \frac{1}{4}\Big\{\Pi^{\gtrless}(X,p), \Delta^+(X,p) + \Delta^-(X,p)\Big\}\\
&- \frac{1}{4}\Big\{\Pi^+(X,p) + \Pi^-(X,p),\, \Delta^{\gtrless}(X,p)\Big\}, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\Big[-p^2 + m^2 &- \Pi_\delta(X)\Big]\Delta^{\gtrless}(X,p)\\
&= \frac{1}{2}\Big(\Pi^{\gtrless}(X,p)\big(\Delta^+(X,p) + \Delta^-(X,p)\big) + \big(\Pi^+(X,p) + \Pi^-(X,p)\big)\Delta^{\gtrless}(X,p)\Big)\\
&+ \frac{i}{4}\Big\{\Pi^>(X,p),\, \Delta^<(X,p)\Big\} - \frac{i}{4}\Big\{\Pi^<(X,p),\, \Delta^>(X,p)\Big\}. \qquad (27)
\end{aligned}
$$

Free transport equation

- In the limit of free fields, Eqs. (26, 27) become

$$
p^\mu\partial_\mu\Delta_0^{\gtrless}(X,p) = 0, \qquad (28)
$$

$$
\big[p^2 - m^2\big]\Delta_0^{\gtrless}(X,p) = 0. \qquad (29)
$$

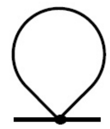

Figure 1: The first order contribution to the self energy

- Due to Eq. (29), $\Delta_0^{\gtrless}(X,p)$ is proportional to $\delta(p^2-m^2)$, and consequently free quasi-particles are always on mass-shell. We note that if the quasi-particle approximation (24) were *not* used, the mass-shell equation would have the form

$$\Big[\frac{1}{4}\partial^2 - p^2 + m^2\Big]\Delta_0^{\gtrless}(X,p) = 0\ , \tag{30}$$

and the off-shell contribution to the Green's function $\Delta_0^{\gtrless}$ would be nonzero.

- Since the free Green's function $\Delta_0^{\gtrless}$, which were derived in Lecture V, are

$$\Delta_0^{>}(X,k) \;=\; -\frac{i\pi}{E_{\mathbf{k}}}\Big[\delta(E_{\mathbf{k}}-k_0)\big(f_0(X,\mathbf{k})+1\big)+\delta(E_{\mathbf{k}}+k_0)\,f_0(X,-\mathbf{k})\Big], \tag{31}$$

$$\Delta_0^{<}(X,k) \;=\; -\frac{i\pi}{E_{\mathbf{k}}}\Big[\delta(E_{\mathbf{k}}-k_0)f_0(X,\mathbf{k})+\delta(E_{\mathbf{k}}+k_0)\big(f_0(X,-\mathbf{k})+1\big)\Big], \tag{32}$$

where $E_{\mathbf{k}} \equiv \sqrt{m^2+\mathbf{k}^2}$, the equation (29) is trivially solved while the transport equation is

$$p^{\mu}\partial_{\mu} f_0(X,\mathbf{p}) = 0. \tag{33}$$

- In the non-convariant notation the transport equation has the familiar form

$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big) f_0(t,\mathbf{r},\mathbf{p}) = 0 \tag{34}$$

where $X=(t,\mathbf{r})$ and $\mathbf{v} \equiv \mathbf{p}/E_{\mathbf{p}}$ is the particle's velocity.

Collisionless transport equation

- To go beyond the free theory, it is necessary to determine the self energies that enter the equations (26, 27). For this purpose, we will refer to the perturbative expansion.
- The first order contribution is shown in Fig. 1 and it is

$$\Pi(x,y) = -i\frac{\lambda}{2}\,\delta^{(4)}(x-y)\,\Delta_0^{<}(x,x), \tag{35}$$

which gives

$$\Pi_{\delta}(X) = -i\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\Delta_0^{<}(X,k). \tag{36}$$

- Keeping in mind the form of the free functions $\Delta_0^{<}(X,k)$ given by Eqs. (31, 32), one obtains

$$\Pi_{\delta}(X) = -\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{f_0(X,\mathbf{k})}{E_k}, \tag{37}$$

where the vacuum contribution has been subtracted. As one observes, $\Pi_{\delta}(X)$ is real and negative.

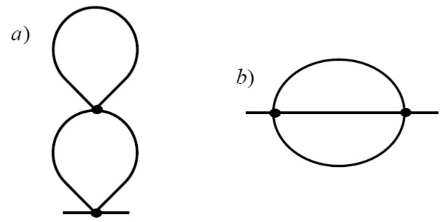

Figure 2: The second order contributions to the self energy

- Since only the one-point self energy $\Pi_\delta(X)$ is nonzero in the first order of perturbative expansion, the transport and mass-shell equations (26, 27) are

$$\Big[p_\mu \partial^\mu - \frac{1}{2}\big(\partial_\mu m_*^2(X)\big)\partial_p^\mu\Big]\Delta^{\gtrless}(X,p) \;=\; 0, \tag{38}$$

$$\Big[-p^2 + m_*^2(X)\Big]\Delta^{\gtrless}(X,p) \;=\; 0, \tag{39}$$

where $m_*(X)$ is the position dependent effective mass defined as

$$m_*^2(X) = m^2 - \Pi_\delta(X). \tag{40}$$

- One sees that the mass-shell equation (39) is satisfied if the Green's functions $\Delta^{\gtrless}(X,p)$ are as in Eq. (31, 32) with the mass m replaced be the effective mass $m_*(X)$.
- Eq. (38) has a form of collisionless or Vlasov transport equation which can be written as

$$\Big[p_\mu \partial^\mu - F_\mu(X)\partial_p^\mu\Big] f(X,\mathbf{p}) = 0, \tag{41}$$

where $F_\mu(X) \equiv \frac{1}{2}\partial_\mu m_*^2(X)$ is the force acting on gas particles.

Collisional transport equation

- The second order contributions to the self energy, which are given by one-particle-irreducible diagrams, are shown in Fig. 2.
- The diagram Fig. 2a provides a correction to the contribution (37). Since it does not introduce a qualitatively new effect, we will not discuss it any more.
- The contribution corresponding to the diagram Fig. 2b is

$$\Pi(x,y) = \frac{\lambda^2}{6}\,\Delta_0(x,y)\,\Delta_0(x,y)\,\Delta_0(y,x), \tag{42}$$

which gives

$$\Pi^{\gtrless}(x,y) = \frac{\lambda^2}{6}\,\Delta_0^{\gtrless}(x,y)\,\Delta_0^{\gtrless}(x,y)\,\Delta_0^{\lessgtr}(y,x). \tag{43}$$

- Performing the Wigner transformation, one finds

$$\Pi^{\gtrless}(X,p) = \frac{\lambda^2}{6}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4r}{(2\pi)^4} \tag{44}$$

$$\times\,(2\pi)^4\delta^{(4)}(p+r-k-q)\,\Delta_0^{\gtrless}(X,k)\,\Delta_0^{\gtrless}(X,q)\,\Delta_0^{\lessgtr}(X,r).$$

- The transport and mass-shell equations (26, 27) are

$$\Big[p^\mu \partial_\mu - \frac{1}{2}\partial_\mu \Pi_\delta(X)\partial_p^\mu\Big]\Delta^{\gtrless}(X,p) = \frac{i}{2}\Big(\Pi^>(X,p)\Delta^<(X,p) - \Pi^<(X,p)\Delta^>(X,p)\Big), \quad (45)$$

$$\Big[-p^2 + m^2 - \Pi_\delta(X)\Big]\Delta^{\gtrless}(X,p) = \frac{1}{2}\Big(\Pi^{\gtrless}(X,p)\big(\Delta^+(X,p) + \Delta^-(X,p)\big) + \big(\Pi^+(X,p) + \Pi^-(X,p)\big)\Delta^{\gtrless}(X,p)\Big), \quad (46)$$

where we have neglected the gradient terms in the right-hand-sides of the equations. These terms, which are of the second order in the perturbative expansion and of the first order in the gradient expansion, are of higher order than the retained terms.

- Since the right-hand-side of equation (46) is non-zero, the Green's functions $\Delta^{\gtrless}(X,p)$ are no longer delta-like. Then, we deal with quasiparticles of finite width or lifetime which seriously complicates an analysis of Eqs. (45, 46). We will ignore the complication, neglecting the right-hand-side of equation (46) which is actually of the higher order in perturbative expansion than the left-hand-side. In such a case, the Green's functions $\Delta^{\gtrless}(X,p)$ are again as in Eq. (31, 32) with the mass m replaced be the effective mass $m_*(X)$ that is

$$\Delta^>(X,k) = -\frac{i\pi}{E_{\mathbf{k}}}\Big[\delta(E_{\mathbf{k}} - k_0)\big(f(X,\mathbf{k}) + 1\big) + \delta(E_{\mathbf{k}} + k_0)\, f(X,-\mathbf{k})\Big], \quad (47)$$

$$\Delta^<(X,k) = -\frac{i\pi}{E_{\mathbf{k}}}\Big[\delta(E_{\mathbf{k}} - k_0) f(X,\mathbf{k}) + \delta(E_{\mathbf{k}} + k_0)\big(f(X,-\mathbf{k}) + 1\big)\Big], \quad (48)$$

where $E_{\mathbf{k}} \equiv \sqrt{m_*^2(X) + \mathbf{k}^2}$.

- Since the functions $\Delta^{\gtrless}(X,p)$ are non-zero only for momenta on mass shell ($p^2 = m_*^2(X)$), the self energies $\Pi^{\gtrless}(X,p)$, which enter the transport equation (45), should be computed for the momenta on mass shell.

- Substituting the functions (47, 48) into the formula (44), one gets 8 terms. If $p_0 = E_{\mathbf{p}}$ five of these eight terms are zero because the corresponding energy conserving delta function has no support. The three non-zero terms can be combined to give

$$\Theta(p_0)\Big(\Pi^>(X,p)\Delta^<(X,p) - \Pi^<(X,p)\Delta^>(X,p)\Big) \quad (49)$$
$$= \frac{\pi\delta(E_{\mathbf{p}} - p_0)}{E_{\mathbf{p}}}\frac{\lambda^2}{2}\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\int\frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}}\int\frac{d^3r}{(2\pi)^3 2E_{\mathbf{r}}}(2\pi)^4\delta^{(4)}(p + r - k - q)$$
$$\times\Big[f_{\mathbf{p}} f_{\mathbf{r}}(f_{\mathbf{k}} + 1)(f_{\mathbf{q}} + 1) - f_{\mathbf{k}} f_{\mathbf{q}}(f_{\mathbf{p}} + 1)(f_{\mathbf{r}} + 1)\Big],$$

where $f_{\mathbf{p}} \equiv f(X,\mathbf{p})$.

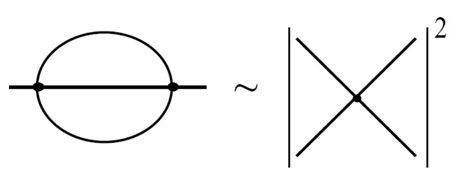

Figure 3: The second order self energy expressed as matrix element squared

- The transport equation (45) becomes

$$\Big[p_\mu \partial^\mu - F_\mu(X)\partial_p^\mu\Big] f(X,\mathbf{p}) = C[f], \tag{50}$$

where the collision term equals

$$C[f] \equiv \frac{\lambda^2}{4} \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \int \frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}} \int \frac{d^3r}{(2\pi)^3 2E_{\mathbf{r}}} (2\pi)^4 \delta^{(4)}(p+r-k-q) \tag{51}$$
$$\times \Big[f_{\mathbf{k}} f_{\mathbf{q}} (f_{\mathbf{p}}+1)(f_{\mathbf{r}}+1) - f_{\mathbf{p}} f_{\mathbf{r}} (f_{\mathbf{k}}+1)(f_{\mathbf{q}}+1) \Big].$$

- The first term in the formula (51) corresponds to the binary process where particles with momenta $\mathbf{k}$ and $\mathbf{q}$ interact with each other and produce particles with momenta $\mathbf{p}$ and $\mathbf{r}$. The second term corresponds to the inverse process. The factors like $f_{\mathbf{k}}+1$ are associated with the Bose-Einstein statistics of particles. In the classical limit $f_{\mathbf{k}} \ll 1$ and $f_{\mathbf{k}}+1 \approx 1$.

- As one observes, the self energies, which enter the transport equation (45), appear in the final collision term (51), as the transition matrix element squared multiplied by the factors like $f_{\mathbf{k}}$ or $f_{\mathbf{k}}+1$. This is expressed graphically in Fig. 3.