Symmetry breaking and effective potential in vacuum QFT

Spontaneous symmetry breaking

- Let us consider a system of real scalar fields with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) + \underbrace{\frac{1}{2}\mu^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x)}_{\equiv -V(\phi)}, \tag{1}$$

  where the mass term has a 'wrong' sign as long as $\mu^2 > 0$.

- One asks what is a ground state or vacuum of such a system. Since the state is assumed to be translationally invariant, the field which minimizes the energy must be a constant. Then, the kinetic term of the Hamiltonian vanishes and the energy is determined by the potential

$$V(\phi) \equiv \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{2}$$

  which is shown in Fig. 1.

- One immediately finds that the potential is minimized by the non-zero field equal

$$\phi_0 = \pm\sqrt{\frac{6\mu^2}{\lambda}}. \tag{3}$$

- To quantize the field one redefines it as

$$\phi(x) \longrightarrow \phi(x) - \phi_0, \tag{4}$$

  and the Lagrangian density (1) changes into

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}\Big(-\mu^2 + \frac{\lambda}{4}\phi_0^2\Big)\phi^2(x) - \frac{\lambda}{3!}\phi_0\phi^3(x) - \frac{\lambda}{4!}\phi^4(x), \tag{5}$$

  where the constant terms and those linear in ϕ are ignored.

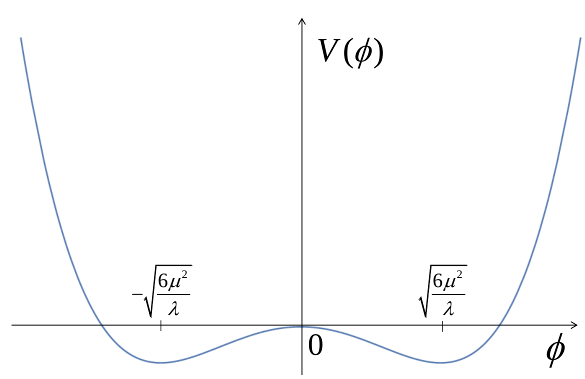

Figure 1: The classical potential of the Lagrangian density (1)

- Substituting ϕ_0 given by Eq. (3) into the Lagrangian (5), one finds

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi(x)\partial_\mu \phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{g}{3!}\phi^3(x) - \frac{\lambda}{4!}\phi^4(x). \tag{6}$$

where

$$m^2 \equiv \frac{1}{2}\mu^2, \qquad\qquad g \equiv \pm\sqrt{6\lambda}\,\mu. \tag{7}$$

- One observes that the Lagrangian density (1) is invariant under the symmetry $\phi \to -\phi$ but the the Lagrangian (6) is not. The choice of one of the two possible ground states leads to the spontaneous symmetry breaking.

- The method to find a ground state which is described above is purely classical. The question arises whether quantum fluctuations of the field do not change the conclusion on the symmetry breaking. The point is that an analog of the classical filed is the quantum field expectation value $\langle\phi\rangle$ and $\langle\phi^2\rangle \neq \langle\phi\rangle^2$, nor $\langle\phi^4\rangle \neq \langle\phi\rangle^4$. A shape of the potential in Fig. 1 suggests that if the ground state energy is increased above the local maximum due to quantum fluctuations there is no symmetry breaking. An effect of thermal fluctuations can be even more dramatic than that of quantum ones.

Effective action

- The expectation value of a quantum field is defined in the Heisenberg picture as

$$\langle\phi(x)\rangle \equiv \frac{\langle 0\,\mathrm{out}|\hat{\phi}(x)|0\,\mathrm{in}\rangle}{\langle 0\,\mathrm{out}|0\,\mathrm{in}\rangle}. \tag{8}$$

- The expectation value is provided by the generating functional $W[J]$ as

$$\langle\phi(x)\rangle = \frac{1}{i}\frac{\partial}{\partial J(x)}W[J]\bigg|_{J=0}. \tag{9}$$

- The generating functional equals

$$W[J] = \mathcal{N}\int \mathcal{D}\phi(x)\exp\Big\{i\Big[S[\phi] + \int d^4x\, J(x)\phi(x)\Big]\Big\}, \tag{10}$$

where the normalization constant $\mathcal{N}$ is

$$\mathcal{N}^{-1} = \int \mathcal{D}\phi(x)\exp\big\{iS[\phi]\big\}. \tag{11}$$

- Actually, the normalization constant $\mathcal{N}$ is of no importance, as further on we will use the generating functional $Z[J]$ of connected Green's function equal

$$Z[J] \equiv -i\ln W[J] \tag{12}$$

and

$$\langle\phi(x)\rangle = \frac{\partial}{\partial J(x)}Z[J]\bigg|_{J=0}. \tag{13}$$

- We are going to develop a formalism where $\langle\phi(x)\rangle$ is treated as a dynamical variable. For this purpose, we define the field expectation value in the presence of J

$$\Phi(x) \equiv \langle\phi(x)\rangle_J = \frac{\partial}{\partial J(x)} Z[J], \tag{14}$$

and we perform the Legendre transformation $Z[J] \to \Gamma[\langle\phi\rangle_J]$ as

$$\Gamma[\Phi] = Z[J] - \int d^4x \, J(x)\Phi(x), \tag{15}$$

which allows one to invert the relation (14) that is to express $J(x)$ through $\Phi(x)$.

- The functional $\Gamma[\Phi]$ is called the <u>effective action</u>. To explain a physical meaning of $\Gamma[\Phi]$, let us compute the derivative

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = \int d^4x' \frac{\delta J(x')}{\delta\Phi(x)} \underbrace{\frac{\delta Z[J]}{\delta J(x')}}_{=\Phi(x')} - \int d^4x' \left(\frac{\delta J(x')}{\delta\Phi(x)} \Phi(x') + J(x') \underbrace{\frac{\delta\Phi(x')}{\delta\Phi(x)}}_{=\delta^{(4)}(x-x')} \right). \tag{16}$$

- Using Eq. (14), we find the key result

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = -J(x). \tag{17}$$

- We note that as long as $J(x) \neq 0$ and $J(x)$ depends on x, the theory is <u>not</u> translationally invariant.

- When $J = 0$, Eq. (17) provides the fundamental relation

$$\boxed{\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = 0,} \tag{18}$$

which is just the reason why $\Gamma[\Phi]$ is called the 'effective action': *the actual field expectation value at $J = 0$ corresponds to the extremum of* $\Gamma[\Phi]$, as the classical solution of equation of motion corresponds to the extremum of $S[\phi]$.

Free effective action

- Let us see how the effective action looks like in the simple case of non-interacting field with the usual mass term. Then, the functional $Z_0[J]$ is

$$Z_0[J] = -\frac{1}{2} \int d^4x \, d^4y J(x) \, \Delta_0(x-y) \, J(y), \tag{19}$$

and

$$\Phi(x) = \frac{\delta Z_0[J]}{\delta J(x)} = -\int d^4y \, \Delta_0(x-y) J(y). \tag{20}$$

We note that even if $\Phi(x)$ vanishes for $J = 0$, $\Phi(x)$ is nonzero for $J \neq 0$.

- The relation (20) can be easily inverted as

$$J(x) = (\Box + m^2)\Phi(x), \tag{21}$$

where one recognizes the equation of motion of $\Phi(x)$.

- Using the formulas (19) and (21), the Legandre transformation (15) can be performed explicitly

$$\begin{aligned}\Gamma_0[\Phi] &= Z_0[J] - \int d^4x\, J(x)\Phi(x) \\ &= -\frac{1}{2}\int d^4x\, d^4y\, \Phi(x)(\overleftarrow{\Box}_x + m^2)\Delta_F(x-y)(\Box_y + m^2)\Phi(y) \\ &\qquad\qquad - \int d^4x\, \Phi(x)(\Box_x + m^2)\Phi(x).\end{aligned} \tag{22}$$

- Applying the Fourier transformation of the fields and Feynman propagator, one shows that

$$\int d^4x\, d^4y\, \Phi(x)(\overleftarrow{\Box}_x + m^2)\Delta_0(x-y)(\Box_y + m^2)\Phi(y) \tag{23}$$
$$= \int \frac{d^4k}{(2\pi)^4}\, \Phi(-k)(k^2 - m^2)\Phi(k) = -\int d^4x\, \Phi(x)(\Box + m^2)\Phi(x),$$

and consequently, the free effective action is

$$\Gamma_0[\Phi] = -\frac{1}{2}\int d^4x\, \Phi(x)(\Box + m^2)\Phi(x). \tag{24}$$

- Performing the partial integration, Eq. (24) can be rewritten as

$$\boxed{\Gamma_0[\Phi] = \int d^4x \left[\frac{1}{2}\partial^\mu\Phi(x)\partial_\mu\Phi(x) - \frac{1}{2}m^2\Phi(x)^2\right] = \int d^4x\, \mathcal{L}_0\big(\Phi(x)\big),} \tag{25}$$

which shows that the free effective action is the free action of $\Phi(x)$.

- The formula (24) can be also written as

$$\Gamma_0[\Phi] = \frac{1}{2}\int d^4x\, d^4y\, \Phi(x)\Delta_0^{-1}(x-y)\Phi(x), \tag{26}$$

where $\Delta_0^{-1}(x-y)$ is the inverse propagator equal

$$\Delta_0^{-1}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}(k^2 - m^2), \tag{27}$$

which obeys

$$\int d^4y\, \Delta_0(x-y)\, \Delta_0^{-1}(y-z) = \delta^{(4)}(x-z). \tag{28}$$

- The formula (26) clearly shows that the effective action generates inverse propagators as

$$\Delta_0^{-1}(x-y) = \frac{\delta^2\Gamma_0[\Phi]}{\delta\Phi(x)\delta\Phi(y)}. \tag{29}$$

One-particle irreducible graphs

- We consider systems where $\Phi = 0$ for $J = 0$. If it happens that $\Phi = v \neq 0$ for $J = 0$, we redefine the field as $\phi(x) \to \phi(x) - v$ and then $\Phi = 0$ at $J = 0$.
- While the functionals $W[J]$ and $Z[J]$ are expanded in powers of J, the effective action can be expanded in powers of Φ in the following way

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 d^4x_2 \dots d^4x_n \, \Gamma^{(n)}(x_1, x_2, \dots x_n) \, \Phi(x_1)\Phi(x_2)\dots\Phi(x_n), \tag{30}$$

where $\Gamma^{(n)}(x_1, x_2, \dots x_n)$ represents one-particle irreducible (1PI) diagrams called *proper vertices* or *vertex functions*.

- *A one-particle irreducible diagram is a connected diagram that cannot be disconnected by cutting a single internal line.*
- The $n-$point vertex function is generated as

$$\Gamma^{(n)}(x_1, x_2, \dots x_n) = \frac{\delta}{\delta\Phi(x_1)} \cdots \frac{\delta}{\delta\Phi(x_n)} \, \Gamma[\Phi]\Big|_{\Phi=0}. \tag{31}$$

- Let us consider one-, two-, three-, and four-point Green's functions to understand why the effective action generates the 1PI diagrams.
- The one-point vertex function is

$$\Gamma^{(1)}(x) = \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\Big|_{\Phi=0} = J(x)\Big|_{\Phi=0} = 0 \tag{32}$$

and it vanishes due to Eq. (17).

- The two-point vertex function is found as

$$\Gamma^{(2)}(x_1, x_2) = \frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x_1)\,\delta\Phi(x_2)}\Big|_{\Phi=0} = -\frac{\delta J(x_2)}{\delta\Phi(x_1)}\Big|_{\Phi=0} = -\left(\frac{\delta\Phi(x_1)}{\delta J(x_2)}\right)^{-1}\Big|_{\Phi=0}, \tag{33}$$

using Eq. (17).

- Since

$$\Phi(x_1) = \frac{\delta Z[J]}{\delta J(x_1)}, \tag{34}$$

we have

$$\Gamma^{(2)}(x_1, x_2) = -\left(\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right)^{-1}\Big|_{\Phi=0} = \Delta^{-1}(x_1, x_2). \tag{35}$$

The two-point full connected Green's function $\Delta_c^{(2)}(x, y)$ is denoted here and further on as $\Delta(x, y)$ with no super- of subscripts.

- The effective action generates the inverse full connected Green's function in complete analogy with the result (29) obtained with the free effective action.

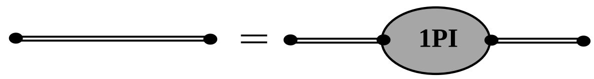

Figure 2: The relation between full connected and 1PI two-point Green's functions

- We note that $\Delta(x,y)$ and $\Delta^{-1}(x,y)$ should be treated as matrices and they obey

$$\int d^4y\, \Delta(x,y)\, \Delta^{-1}(y,z) = \delta^{(4)}(x-z), \tag{36}$$

which changes into Eq. (28) in case of a translationally invariant system.

- Eq. (35) can be written as

$$\Delta(y_1,y_2) = \int d^4x_1\, d^4x_2\, \Delta(y_1,x_1)\, \Gamma^{(2)}(x_1,x_2)\, \Delta(x_2,y_2), \tag{37}$$

and it is represented graphically in Fig. 2. The equation just means that the full connected propagator is obtained from 1PI two-point function $\Gamma^{(2)}$ by adding two external legs with propagators.

- From Eqs. (35) and (35) one can deduce the identity

$$\frac{\delta}{\delta\Phi(x)} = \int d^4y\, \frac{\delta J(y)}{\delta\Phi(x)}\frac{\delta}{\delta J(y)} = \int d^4y\, \Delta_J^{-1}(x,y)\, \frac{\delta}{\delta J(y)}, \tag{38}$$

where $\Delta_J(x,y)$ denotes the two-point connected Green's function the presence of $J \neq 0$. Eq. (38) explains why the effective action generates 1PI graphs: when the derivative $\delta/\delta J$ acts on $Z[J]$ it produces the external line with the propagator. When the derivative $\delta/\delta\Phi$ acts on $Z[J]$ the propagator is removed. The two subsequent examples show hot it works.

- The three-point vertex function equals

$$\Gamma^{(3)}(x_1,x_2,x_3) = \frac{\delta^3\Gamma[\Phi]}{\delta\Phi(x_1)\,\delta\Phi(x_2)\,\delta\Phi(x_3)}\bigg|_{\Phi=0} = -\frac{\delta}{\delta\Phi(x_1)}\left(\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_3)}\right)^{-1}\bigg|_{\Phi=0}$$

$$= -\int d^4y_1\, \Delta_J^{-1}(x_1,y_1)\, \frac{\delta}{\delta J(y_1)}\left(\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_3)}\right)^{-1}\bigg|_{\Phi=0}, \tag{39}$$

where we first used the result (35) and then the identity (38).

- To compute the derivative $\delta/\delta J(y_1)$, one must remember that

$$\left(\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_3)}\right)^{-1} = \Delta_J^{-1}(x_2,x_3) \tag{40}$$

and that $\Delta_J^{-1}(x_2,x_3)$ should be treated as a matrix. Then,

$$\frac{\delta}{\delta J(y_1)}\left(\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_3)}\right)^{-1} = \int d^4y_2\, d^4y_3\, \Delta_J^{-1}(x_2,y_2)\, \frac{\delta^3 Z[J]}{\delta J(y_1)\delta J(y_2)\delta J(y_3)}\, \Delta_J^{-1}(y_3,x_3). \tag{41}$$

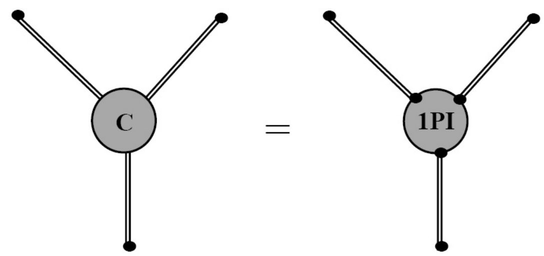

Figure 3: The relation between full connected and 1PI three-point Green's functions

- Substituting the result (41) into Eq. (39), we get

$$\begin{aligned}\Gamma^{(3)}(x_1,x_2,x_3) &= \int d^4y_1\, d^4y_2\, d^4y_3\, \Delta_J^{-1}(x_1,y_1)\, \Delta_J^{-1}(x_2,y_2)\, \Delta_J^{-1}(y_3,x_3) \\ &\qquad\qquad \times \left.\frac{\delta^3 Z[J]}{\delta J(y_1)\delta J(y_2)\delta J(y_3)}\right|_{\Phi=0} \\ &= \int d^4y_1\, d^4y_2\, d^4y_3\, \Delta^{-1}(x_1,y_1)\, \Delta^{-1}(x_2,y_2)\, \Delta^{-1}(y_3,x_3)\, \Delta_c^{(3)}(y_1,y_2,y_3).\end{aligned} \tag{42}$$

- The relation (42), which can be rewritten as

$$\Delta_c^{(3)}(y_1,y_2,y_3) = \int d^4x_1\, d^4x_2\, d^4x_3\, \Delta(y_1,x_1)\, \Delta(y_2,x_2)\, \Delta(y_3,x_3)\, \Gamma^{(3)}(x_1,x_2,x_3), \tag{43}$$

  is represented graphically in Fig. (3).

- The four-point vertex function equals

$$\begin{aligned}\Gamma^{(4)}(x_1,x_2,x_3,x_4) &= \left.\frac{\delta^4\Gamma[\Phi]}{\delta\Phi(x_1)\,\delta\Phi(x_2)\,\delta\Phi(x_3)\,\delta\Phi(x_4)}\right|_{\Phi=0} \\ &= \int d^4y_1\, d^4y_2\, d^4y_3\, d^4y_4\, \Delta_J^{-1}(x_1,y_1)\, \frac{\delta}{\delta J(y_1)}\, \Delta_J^{-1}(x_2,y_2)\, \Delta_J^{-1}(y_3,x_3)\, \Delta_J^{-1}(y_4,x_4) \\ &\qquad\qquad \times \left.\frac{\delta^3 Z[J]}{\delta J(y_2)\delta J(x_3)\delta J(x_4)}\right|_{\Phi=0}.\end{aligned} \tag{44}$$

- One observes that taking the derivative in Eq. (44), we will produce four terms: three with the product $\Delta_c^{(3)}\Delta_c^{(3)}$ and one with $\Delta_c^{(4)}$. The computation is very tedious but the final result can be easily guessed. It is represented in Fig. 4.

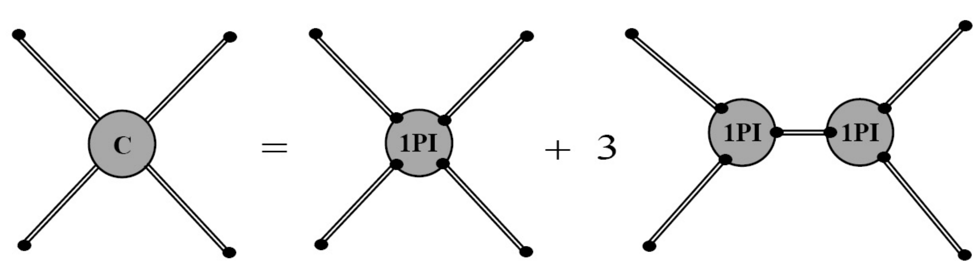

Figure 4: The relation between full connected and 1PI four-point Green's functions

- The analysis of the above examples shows that the n-point vertex function $\Gamma^{(n)}$ is obtained from the n-point connected function $\Delta_c^{(n)}$ by removing the external lines and by subtracting the diagrams with one internal line connecting two subdiagrams. (The later aspect is clearly seen in Fig. 4.) This just explains why $\Gamma^{(n)}$ is represented by one-particle-irreducible diagrams. A general proof can be done by induction.

Action of 'shifted field'

- Let us consider the generating functional of the 'shifted field' $\phi \to \phi + \phi_0$, which is

$$\widetilde{W}[J, \phi_0] = \int \mathcal{D}\phi(x) \exp\Big\{i\Big[S[\phi + \phi_0] + \int d^4x\, J(x)\phi(x)\Big]\Big\}, \tag{45}$$

where the normalization constant is ignored, as it plays no role here.

- Changing the variable of functional integration $\phi \to \phi + \phi_0$, one finds

$$\begin{aligned} \widetilde{W}[J, \phi_0] &= \int \mathcal{D}\phi(x) \exp\Big\{i\Big[S[\phi] + \int d^4x\, J(x)\big(\phi(x) - \phi_0(x)\big)\Big]\Big\} \\ &= \exp\Big[-i\int d^4x\, J(x)\phi_0(x)\Big] \int \mathcal{D}\phi(x) \exp\Big\{i\Big[S[\phi] + \int d^4x\, J(x)\phi(x)\Big]\Big\} \\ &= W[J] \exp\Big[-i\int d^4x\, J(x)\phi_0(x)\Big]. \end{aligned} \tag{46}$$

- Now, one defines the generating functional of connected diagrams

$$\widetilde{Z}[J, \phi_0] \equiv -i \ln \widetilde{W}[J, \phi_0] = Z[J] - \int d^4x\, J(x)\phi_0(x). \tag{47}$$

- Comparing Eq. (47) with the effective action definition (15), one realizes that

$$\Gamma[\phi_0]\Big|_{J=0} = \widetilde{Z}[J = 0, \phi_0] = -i \ln\Big[\int \mathcal{D}\phi(x) \exp\Big(iS[\phi + \phi_0]\Big)\Big], \tag{48}$$

which actually suggests a method to compute the effective action starting with the action of shifted field $S[\phi + \phi_0]$.

Effective potential

- In a translationally invariant theory, that is with $J = 0$, the field expectation value $\Phi(x)$ is independent of x which greatly simplifies the problem of finding a ground state of the theory.

- The ground state is

$$\bar{\phi}_{\rm min} = \langle\phi\rangle, \tag{49}$$

and it is found as a solution of the fundamental equation (18), which for translationally invariant system becomes

$$\frac{\partial\Gamma[\bar{\phi}]}{\partial\bar{\phi}} = 0, \tag{50}$$

where we can use the usual not functional derivative. It is understood here and further on that $J = 0$.

- The effective action also simplifies. Since $\bar{\phi}$ is independent of x, the kinetic term of the action vanishes and

$$\Gamma[\bar{\phi}] = -\int d^4x\, V_{\rm eff}(\bar{\phi}) = -V\mathcal{T}\, V_{\rm eff}(\bar{\phi}), \tag{51}$$

which is just the definition of *effective potential* with

$$V\mathcal{T} \equiv \int d^4x. \tag{52}$$

- The vacuum state of the theory is determined by the equation

$$\boxed{\frac{\partial V_{\rm eff}(\bar{\phi})}{\partial\bar{\phi}} = 0,} \tag{53}$$

that is as extremuum of the effective potential.

- The expansion of $\Gamma[\Phi]$ in powers of Φ, which is given the formula (30), is greatly simplified when $\Phi = \bar{\phi}$ is a constant. To derive a respective formula, we rewrite the expansion (30) in terms of Fourier transformed fields and vertices

$$\begin{aligned}\Gamma[\Phi] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1 d^4x_2\dots d^4x_n \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\int \frac{d^4p'_1}{(2\pi)^4}\frac{d^4p'_2}{(2\pi)^4}\cdots\frac{d^4p'_n}{(2\pi)^4} \\ \times e^{-i(p_1+p'_1)x_1}e^{-i(p_2+p'_2)x_2}\dots e^{-i(p_n+p'_n)x_n} \\ \times (2\pi)^4\delta^{(4)}(p_1+p_2+\dots+p_n)\,\Gamma^{(n)}(p_1,p_2,\dots,p_n)\,\Phi(p'_1)\,\Phi(p'_2)\dots\Phi(p'_n) \\ = \sum_{n=0}^{\infty}\frac{1}{n!}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\dots+p_n) \\ \times \Gamma^{(n)}(p_1,p_2,\dots,p_n)\,\Phi(-p_1)\,\Phi(-p_2)\dots\Phi(-p_n),\end{aligned} \tag{54}$$

where we have taken into account that position variables $x_1, x_2, \dots, x_n$ of a vertex function $\Gamma^{(n)}(x_1, x_2, \dots, x_n)$ are not independent of each other in a translationally invariant system, because the vertex function must be independent of $X \equiv \frac{1}{n}(x_1+x_2+\dots+x_n)$ which remains unchanged at translations. This fact is taken into account by including the delta function $\delta^{(4)}(p_1+p_2+\dots+p_n)$ in Eq. (54).

- Since the Fourier transform of the constant field $\bar{\phi}$ equals

$$\int d^4x\, e^{ipx}\,\bar{\phi} = (2\pi)^4\delta^{(4)}(p)\,\bar{\phi}, \tag{55}$$

all momentum integrations in Eq. (54) become trivial and

$$\Gamma[\bar{\phi}] = \sum_{n=0}^{\infty}\frac{1}{n!}(2\pi)^4\delta^{(4)}(p=0)\,\Gamma^{(n)}(0,0,\dots,0)\,\bar{\phi}^n = V\mathcal{T}\sum_{n=0}^{\infty}\frac{1}{n!}\Gamma^{(n)}(0,0,\dots,0)\,\bar{\phi}^n, \tag{56}$$

where the infinite factor $(2\pi)^4\delta^{(4)}(p=0)$ is identified with the space-time volume $V\mathcal{T}$.

- Comparing the equations (51) and (56), the expansion of effective potential is found to be

$$V_{\rm eff}(\bar{\phi}) = \sum_{n=0}^{\infty}\frac{1}{n!}\Gamma^{(n)}(0,0,\dots,0)\,\bar{\phi}^n, \tag{57}$$

which says that the effective potential is given by one-particle irreducible vertices with vanishing external momenta.

Expansion in loops

- One wonders how to compute the effective potential. Since the field expectation value does not need to be 'small', it is hard to expect that the first few terms of the expansion (57) will provide a reasonable result. One should also remember that the vertex functions, which enter the series (57), require, in principle, an infinite summation of Feynman diagrams. So, it is important to find a sensible approximate method for a computation of V_{eff}.
- As we remember, the whole idea of the effective action and effective potential was formulated to go beyond the classical method of find a vacuum state of quantum fields. This suggests that a computation of V_{eff} should be based on a quasi-classical approximation. In other words, the Planck constant should be treated as a small parameter. It turns out that a loop expansion of Feynman diagrams is the right approach. Let us explain how it happens.
- First all, we reinstate the Planck constant $\hbar$ into our considerations through the replacement

$$S \to \frac{S}{\hbar} = \frac{1}{\hbar}\int d^4x\, \mathcal{L} = \int d^4x \left[\frac{1}{2\hbar}\Big(\partial^\mu\phi\, \partial_\mu\phi - m^2\phi^2\Big) - \frac{\lambda}{4!\,\hbar}\,\phi^4\right]. \tag{58}$$

- One asks how the theory changes, when the Lagrangian is divided by $\hbar$. Keeping in mind that the free propagator is inversely proportional to m^2 at vanishing four-momentum, one realizes that the new propagator is $\hbar\Delta$ while the new coupling is $\lambda/\hbar$.
- A contribution corresponding to the graph with I internal lines and V vertices is proportional to $\hbar^{I-V}$. It is important here that we deal with 1PI graphs, for which there are no propagators attached to external lines.
- One observes that the number of loops L in a graph with I internal lines and V vertices is $L = I - V + 1$ and thus $I - V = L - 1$. Considering a few examples of diagrams with different number of loops is helpful to understand the relations.
- A contribution corresponding to the graph with L loops is proportional to $\hbar^{L-1}$.
- Since there is an over-all factor $\hbar$, which multiplies the diagram to produce a contribution to the effective potential of the right dimension, the final result is that a contribution corresponding to the graph with L loops is proportional to $\hbar^L$.
- The tree graph with no loop corresponds to the classical potential with no $\hbar$, the one-loop graph is proportional to $\hbar$ and it provides the first quantum correction, etc.

Computation of effective potential

- As we already know, a perturbative expansion of $\Gamma^{(n)}$ can be obtained from the expansion of $\Delta_c^{(n)}$ by amputating external lines and keeping only one-particle irreducible diagrams. The vertices which remain after the amputation of external lines are connected to the fields $\bar\phi$.
- Further on, we compute the effective potential in two cases: when a symmetry breaking is not expected and when a spontaneous symmetry breaking occurs in a classical approach. In the later case the symmetry breaking in the quantum approach is an open question.

Symmetric system

- Let us first discuss a system of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) \underbrace{-\frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x)}_{\equiv -V(\phi)}, \tag{59}$$

which is not expected to experience a spontaneous symmetry breaking.

- To identify the vertices to be used in the computation of the effective potential, we consider the Lagrangian which enters the action (48) of 'shifted field'. The Lagrangian reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\big(\phi(x)+\bar{\phi}\big)^2 - \frac{\lambda}{4!}\big(\phi(x)+\bar{\phi}\big)^2 \\ &= -\frac{1}{2}m^2\bar{\phi}^2 - \frac{\lambda}{4!}\,\bar{\phi}^4 + \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) \\ &\quad + \big(m^2\bar{\phi} + \frac{\lambda}{6}\,\bar{\phi}^3\big)\phi(x) - \frac{\lambda}{4}\,\bar{\phi}^2\phi^2(x) - \frac{\lambda}{6}\,\bar{\phi}\phi^3(x) - \frac{\lambda}{24}\,\phi^4(x). \end{aligned} \tag{60}$$

- The zero-loop vertex functions, which occur due to the mass ($\frac{1}{2}m^2\bar{\phi}^2$) and interaction ($\frac{\lambda}{4!}\,\bar{\phi}^4$) terms of the Lagrangian (60), are

$$\Gamma^{(2)}_{0-\text{loop}}(0,0) = 2!\,\frac{1}{2}\,m^2 = m^2, \qquad \Gamma^{(4)}_{0-\text{loop}}(0,0,0,0) = 4!\,\frac{\lambda}{4!} = \lambda, \tag{61}$$

where the combinatoric factors 2! and 4! reflect in how many ways two or four fields $\bar{\phi}$ can be attached to the two- and four-point vertices.

- Substituting the vertices (61) into the expansion (57) we get

$$V^{0-\text{loop}}_{\text{eff}}(\bar{\phi}) = \frac{1}{2}m^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4, \tag{62}$$

which is just the classical potential.

- The one-loop vertex function $\Gamma^{(2)}$ of the first order in λ is represented by the first graph in Fig. 5 and it occurs due to the interaction ($\frac{\lambda}{4}\,\bar{\phi}^2\phi^2$) term of the Lagrangian (60). We note that the amputated external lines would be connected to the fields $\bar{\phi}$. The function reads

$$\Gamma^{(2)}_{1-\text{loop}}(0,0) = -i\,\frac{i\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\,i\Delta(k) = i\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i0^+}. \tag{63}$$

The over-all factor $-i$ comes from the definition (12) of $Z[J]$. The included combinatoric factor 2 reflects the number of ways in which two fields $\bar{\phi}$ are attached to the vertex.

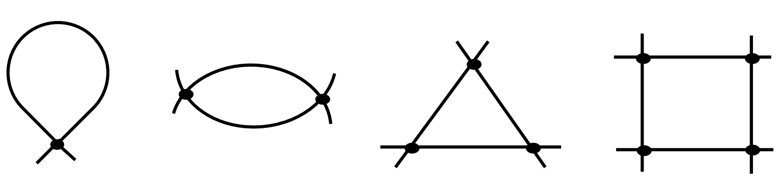

Figure 5: The one-loop diagrams

- The one-loop vertex functions $\Gamma^{(4)}$, $\Gamma^{(6)}$ and $\Gamma^{(8)}$, which are of the order λ^2, λ^3 and λ^4, are represented by the graphs in Fig. 5. All these contributions to the effective potential are due to the interaction term $\frac{\lambda}{4}\,\bar{\phi}^2\phi^2$.

- One realizes that the one-loop vertex function $\Gamma^{(2n)}$ of the order λ^n equals

$$\Gamma^{(2n)}_{1-\text{loop}}(0,0\ldots,0) = i\frac{(n-1)!}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{\frac{1}{2}\lambda}{k^2-m^2+i0^+}\right)^n, \tag{64}$$

  which is a generalization of the result (63). The factor $\frac{1}{2}(n-1)!$ gives the number of ways to order n vertices along the loop.

- We note that the one-loop diagram of order λ^n discussed here is fully analogous to the ring or daisy diagram analyzed in Lecture XIII.

- Since the vertex functions $\Gamma^{(k)}$ (64) are nonzero only for even number $k=2n$, the expansion (57) is effectively not in $\bar{\phi}$ but in $\bar{\phi}^2$. Then, substituting the vertices (64) into the modified expansion, one obtains

$$V^{1-\text{loop}}_{\text{eff}}(\bar{\phi}) = i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\,\bar{\phi}^2}{k^2-m^2+i0^+}\right)^n. \tag{65}$$

- Keeping in mind the Taylor expansion of the logarithm, which is

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty}\frac{x^n}{n}, \tag{66}$$

  the summation in Eq. (65) can be performed and the result is

$$V^{1-\text{loop}}_{\text{eff}}(\bar{\phi}) = -\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\,\ln\left(1-\frac{\frac{1}{2}\lambda\,\bar{\phi}^2}{k^2-m^2+i0^+}\right). \tag{67}$$

- Our next step is called *a Wick rotation.* We rotate the integration path in the plane of complex k^0. Instead of integrating over k^0 from $-\infty$ to ∞ along the real axis, we integrate from $-i\infty$ to $i\infty$ along the imaginary axis, see Fig. 6. Introducing the new variable k^4 such that $k^0 = ik^4$, we integrate over k^4, which is real, from $-\infty$ to ∞.

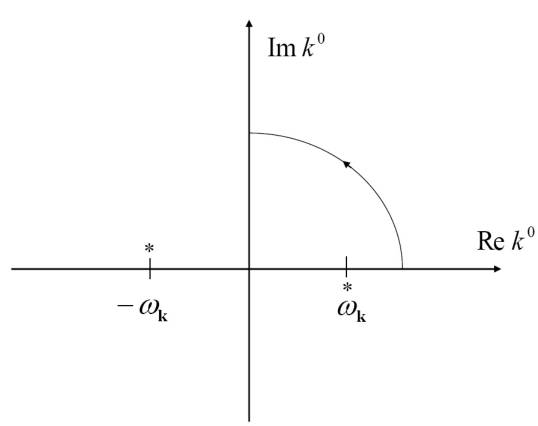

Figure 6: The Wick rotation

- We note that due to the position of the poles of the Feynman propagator, the Wick rotation is made without crossing either pole, as shown in Fig. 6. Therefore, the integral over k_0 from $-\infty$ to ∞ supplemented with the integral along the big half circle does not change its value because of the rotation.

- After the Wick rotation the integral (67) is taken over the Euclidean momentum space and it reads
$$V_{\rm eff}^{1-{\rm loop}}(\bar{\phi}) = \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\,\ln\left(1+\frac{\frac{1}{2}\lambda\,\bar{\phi}^2}{k_E^2+m^2}\right), \tag{68}$$
where $d^4k_E = dk^1dk^2dk^3dk^4$ and $k_E^2 = (p^1)^2+(p^2)^2+(p^3)^2+(p^4)^2$.

- Since $k_E^2+m^2$ is positive for any $k_E = (k^4,\mathbf{k})$, the infinitesimal imaginary element $i0^+$ plays no role and it is ignored.

- The integral (68) is computed in the four-dimensional spherical coordinates as
$$V_{\rm eff}^{1-{\rm loop}}(\bar{\phi}) = \frac{1}{2(2\pi)^4}\int d^3\Omega\int_0^\Lambda dk\,k^3\,\ln\left(1+\frac{a}{k^2+m^2}\right), \tag{69}$$
where $a \equiv \frac{1}{2}\lambda\,\bar{\phi}^2$ and the angular integral, as we remember, equals
$$\int d^3\Omega = 2\pi^2, \tag{70}$$
and $k \equiv \sqrt{k_E^2}$. Since the integral is ultraviolet divergent, the upper cut-off is introduced in Eq. (69).

- Performing the partial integration, one finds
$$\begin{aligned} V_{\rm eff}^{1-{\rm loop}}(\bar{\phi}) &= \frac{1}{2^4\pi^2}\int_0^\Lambda dk\,\frac{1}{4}\frac{dk^4}{dk}\,\ln\left(1+\frac{a}{k^2+m^2}\right) \\ &= \frac{1}{2^6\pi^2}\left[\Lambda^4\ln\left(1+\frac{a}{\Lambda^2+m^2}\right)+2a\int_0^\Lambda\frac{dk\,k^5}{(k^2+m^2+a)(k^2+m^2)}\right]. \end{aligned} \tag{71}$$

- Since $\Lambda^2 \gg m^2$ and $\Lambda^2 \gg a \equiv \frac{1}{2}\lambda\,\bar{\phi}^2$, we approximate the logarithm from Eq. (71) as
$$\Lambda^4\ln\left(1+\frac{a}{\Lambda^2+m^2}\right) \approx \Lambda^4\ln\left(1+\frac{a}{\Lambda^2}\right) \approx a\Lambda^2 - \frac{1}{2}\,a^2 - am^2, \tag{72}$$
where the terms, which vanish in the limit $\Lambda\to\infty$, are neglected.

- Now, let us compute the elementary integral from Eq. (71). With the variable $x \equiv k^2+m^2$, the integral becomes
$$\begin{aligned} I \equiv \int_0^\Lambda \frac{dk\,k^5}{(k^2+m^2+\frac{1}{2}\lambda\,\bar{\phi}^2)(k^2+m^2)} &= \frac{1}{2}\int_{m^2}^{\Lambda^2+m^2}\frac{dx(x-m^2)^2}{(x+a)x} \\ &= \frac{1}{2}\int_{m^2}^{\Lambda^2+m^2}\frac{dx\,x}{x+a} - m^2\int_{m^2}^{\Lambda^2+m^2}\frac{dx}{x+a} + \frac{m^4}{2}\int_{m^2}^{\Lambda^2+m^2}\frac{dx}{(x+a)x}. \end{aligned} \tag{73}$$

- Since

$$\int_{m^2}^{\Lambda^2+m^2} \frac{dx\, x}{x+a} = \Lambda^2 - a\ln\Big(\frac{\Lambda^2+m^2+a}{m^2+a}\Big), \tag{74}$$

$$\int_{m^2}^{\Lambda^2+m^2} \frac{dx}{x+a} = \ln\Big(\frac{\Lambda^2+m^2+a}{m^2+a}\Big), \tag{75}$$

$$\int_{m^2}^{\Lambda^2+m^2} \frac{dx}{(x+a)x} = -\frac{1}{a}\ln\Big(\frac{\Lambda^2+m^2+a}{m^2+a}\Big) + \frac{1}{a}\ln\Big(\frac{\Lambda^2+m^2}{m^2}\Big), \tag{76}$$

one finds

$$I = \frac{1}{2}\Lambda^2 - \frac{1}{2a}\big(m^2+a\big)^2 \ln\Big(\frac{\Lambda^2+m^2+a}{m^2+a}\Big) + \frac{m^4}{2a}\ln\Big(\frac{\Lambda^2+m^2}{m^2}\Big). \tag{77}$$

- Because $\Lambda^2 \gg m^2$ and $\Lambda^2 \gg a \equiv \frac{1}{2}\lambda\,\bar{\phi}^2$, we can use the approximation

$$I \approx \frac{1}{2}\Lambda^2 - \frac{1}{2a}\big(m^2+a\big)^2 \ln\Big(\frac{\Lambda^2}{m^2+a}\Big) + \frac{m^4}{2a}\ln\Big(\frac{\Lambda^2}{m^2}\Big). \tag{78}$$

- Substituting the results (72) and (78) into Eq. (71), we obtain

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}) = \frac{1}{2^6\pi^2}\Big[2a\Lambda^2 - \frac{1}{2}\,a^2 - am^2 - \big(m^2+a\big)^2\ln\Big(\frac{\Lambda^2}{m^2+a}\Big) + m^4\ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big]. \tag{79}$$

- Combining the zero- and one-loop results (62) and (79), the effective potential becomes

$$\begin{aligned} V_{\text{eff}}(\bar{\phi}) = {} & \frac{1}{2}m^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 + \frac{1}{2}\delta m^2\bar{\phi}^2 + \frac{\delta\lambda}{4!}\,\bar{\phi}^4 + \frac{\lambda\Lambda^2}{64\pi^2}\,\bar{\phi}^2 - \frac{\lambda m^2}{128\pi^2}\,\bar{\phi}^2 \\ & - \frac{\lambda^2}{512\pi^2}\bar{\phi}^4 - \frac{1}{64\pi^2}\Big(m^2+\frac{1}{2}\lambda\,\bar{\phi}^2\Big)^2\ln\Big(\frac{\Lambda^2}{m^2+\frac{1}{2}\lambda\,\bar{\phi}^2}\Big) + \frac{m^4}{64\pi^2}\ln\Big(\frac{\Lambda^2}{m^2}\Big), \end{aligned} \tag{80}$$

where we have included the mass and coupling constant counterterms to implement a renormalization procedure which allows one to eliminate the dependence of the effective potential on the cut-off parameter Λ.

- We adopt the following renormalization conditions

$$\frac{d^2V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^2}\bigg|_{\bar{\phi}=0} = m^2, \tag{81}$$

$$\frac{d^4V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^4}\bigg|_{\bar{\phi}=0} = \lambda. \tag{82}$$

- Since

$$\frac{d^2V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^2}\bigg|_{\bar{\phi}=0} = m^2 + \delta m^2 + \frac{\lambda}{32\pi^2}\Big(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\Big), \tag{83}$$

the condition (81) gives

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\Big(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\Big). \tag{84}$$

- Because
$$\frac{d^4 V_{\rm eff}(\bar\phi)}{d\bar\phi^4}\bigg|_{\bar\phi=0} = \lambda + \delta\lambda - \frac{3\lambda^2}{32\pi^2}\Big(\ln\frac{\Lambda^2}{m^2} - 1\Big), \tag{85}$$
the condition (82) provides
$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\Big(\ln\frac{\Lambda^2}{m^2} - 1\Big). \tag{86}$$

- Let us note that while the mass counterterm is of order λ, the coupling constant counterterm is of order λ^2.

- Substituting the results (84) and (86) into Eq. (80), one finds
$$V_{\rm eff}(\bar\phi) = \frac{1}{2}m^2\bar\phi^2 + \frac{\lambda}{4!}\bar\phi^4 - \frac{\lambda}{64\pi^2}\Big(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\Big)\bar\phi^2 + \frac{\lambda^2}{256\pi^2}\Big(\ln\frac{\Lambda^2}{m^2} - 1\Big)\bar\phi^4 + \frac{\lambda\Lambda^2}{64\pi^2}\,\bar\phi^2$$
$$- \frac{\lambda m^2}{128\pi^2}\,\bar\phi^2 - \frac{\lambda^2}{512\pi^2}\bar\phi^4 - \frac{1}{64\pi^2}\Big(m^2 + \frac{1}{2}\lambda\,\bar\phi^2\Big)^2 \ln\Big(\frac{\Lambda^2}{m^2 + \frac{1}{2}\lambda\,\bar\phi^2}\Big) + \frac{m^4}{64\pi^2}\ln\Big(\frac{\Lambda^2}{m^2}\Big), \tag{87}$$
which is manipulated to the form
$$\begin{aligned} V_{\rm eff}(\bar\phi) &= \frac{1}{2}m^2\bar\phi^2 + \frac{\lambda}{4!}\bar\phi^4 \\ &\quad + \frac{1}{64\pi^2}\Big[\Big(m^2 + \frac{1}{2}\lambda\,\bar\phi^2\Big)^2 \ln\Big(1 + \frac{\lambda\,\bar\phi^2}{2m^2}\Big) - \frac{\lambda}{2}\,m^2\,\bar\phi^2 - \frac{3\lambda^2}{8}\,\bar\phi^4\Big], \end{aligned} \tag{88}$$
where the cut-off parameter Λ is absent.

- The one-loop contribution to the effective potential does not make any significant change when compared with the classical potential. The potential has a minimum at $\bar\phi = 0$.

Mass as an interaction vertex

- The structure of our final result (88) suggests a reorganization of the perturbative expansion in such a way that the free theory is treated as massless but the mass term is included in the interaction. Then, the one-loop contribution to the effective potential is due to the interaction term
$$\frac{1}{2}\big(m^2 + \frac{\lambda}{2}\,\bar\phi^2\big)\phi^2. \tag{89}$$

- Using the propagator of the massless field, instead of Eq. (68) one obtains
$$V_{\rm eff}^{1-{\rm loop}}(\bar\phi) = \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\,\ln\Big(1 + \frac{m^2 + \frac{1}{2}\lambda\,\bar\phi^2}{k_E^2}\Big). \tag{90}$$

- Further calculations are fully analogous to those which lead us from Eq. (68) to the effective potential (88). The only difference is that in the final potential there are extra terms independent of $\bar\phi$ which can be simply ignored.

<u>Exercise</u>: Derive the effective potential using the formula (90).

The third method to compute the effective potential

- One rewrites the integral (68) as

$$V_{\rm eff}^{1-{\rm loop}}(\bar\phi) = \frac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\Big(\frac{k_E^2+m^2+\frac{1}{2}\lambda\,\bar\phi^2}{k_E^2+m^2}\Big) \tag{91}$$
$$= \frac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\Big[\ln\Big(\frac{k_E^2+m^2+\frac{1}{2}\lambda\,\bar\phi^2}{m^2}\Big) - \ln\Big(\frac{k_E^2+m^2}{m^2}\Big)\Big].$$

- Since the second term in the formula (91) is independent of $\bar\phi$, it can be simply ignored and the one-loop contribution to the effective potential is

$$V_{\rm eff}^{1-{\rm loop}}(\bar\phi) = \frac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\Big(\frac{k_E^2+m^2+\frac{1}{2}\lambda\,\bar\phi^2}{m^2}\Big). \tag{92}$$

- Further calculations are fully analogous to those which lead us from Eq. (68) to the effective potential (88). The only difference is that in the final potential there are extra terms independent of $\bar\phi$ which can be simply ignored.

Exercise: Derive the effective potential using the formula (92).

Symmetry breaking expected

- We return to the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) + \frac{1}{2}\mu^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \tag{93}$$

where the mass term has a 'wrong' sign $\mu^2 > 0$ which causes, at least at a classical level, the spontaneous symmetry breaking.

- As we already know, the mass term which enters the Lagrangian can be treated as an interaction when the effective potential is computed. Therefore, it does not make a difference whether the term is positive or negative.

- Instead of the renormalization condition (81), we now use

$$\frac{d^2V_{\rm eff}(\bar\phi)}{d\bar\phi^2}\bigg|_{\bar\phi=0} = -\mu^2. \tag{94}$$

However, μ^2 is not a particle's mass but a *physical* parameter of the theory.

- The renormalization condition, which determines the coupling constant, remains Eq. (82).

- The effective potential is given by the formula (88) with m^2 replaced by $-\mu^2$ that is

$$V_{\rm eff}(\bar\phi) = -\frac{1}{2}\mu^2\bar\phi^2 + \frac{\lambda}{4!}\bar\phi^4 \tag{95}$$
$$+\frac{1}{64\pi^2}\Big[\Big(-\mu^2+\frac{1}{2}\lambda\,\bar\phi^2\Big)^2\ln\Big(1-\frac{\lambda\,\bar\phi^2}{2\mu^2}\Big) + \frac{1}{2}\,\lambda\mu^2\,\bar\phi^2 - \frac{3\lambda^2}{8}\,\bar\phi^4\Big].$$

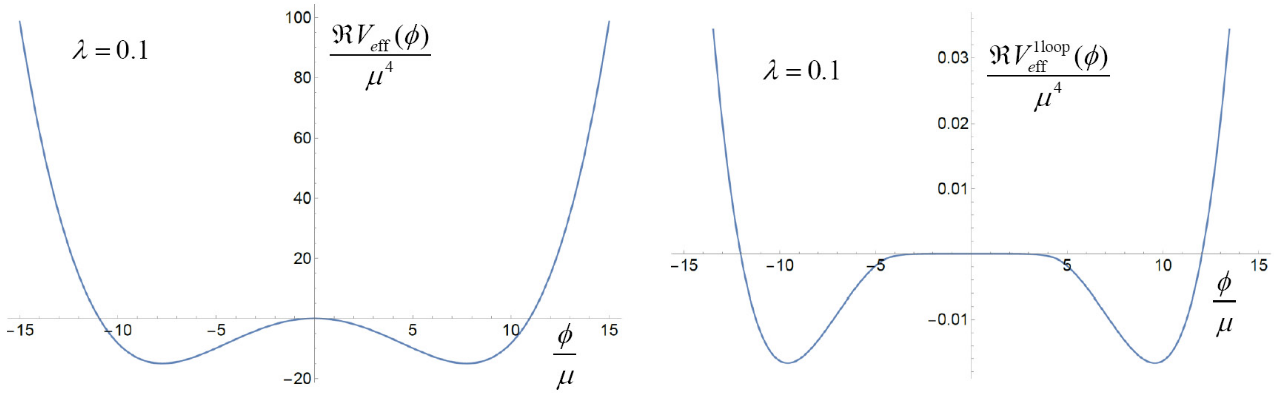

Figure 7: The real part of the complete effective potential (99) (left panel) and the real part of the one-loop contribution to the effective potential (99) (right panel).

- One observes that the logarithm, which enters the formula (95), becomes complex when

$$\bar{\phi}^2 > \frac{2\mu^2}{\lambda}. \tag{96}$$

- As we remember, a logarithm of negative, or more generally, of complex argument is a multivalued function. When the argument z is expressed as

$$z = |z|e^{i(\alpha+2\pi n)}, \qquad n = 0, \pm 1, \pm 2 \dots \tag{97}$$

  the logarithm of z equals

$$\ln z = \ln |z| + i(\alpha + 2\pi n). \tag{98}$$

- Choosing the *principle argument* as $-\pi \leq \alpha < \pi$, the real and imaginary parts of the effective potential (95) read

$$\begin{aligned} \Re V_{\text{eff}}(\bar{\phi}) &= -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 \\ &\quad + \frac{1}{64\pi^2}\Big[\Big(-\mu^2 + \frac{1}{2}\lambda\,\bar{\phi}^2\Big)^2 \ln\Big|1 - \frac{\lambda\,\bar{\phi}^2}{2\mu^2}\Big| + \frac{1}{2}\,\lambda\mu^2\,\bar{\phi}^2 - \frac{3\lambda^2}{8}\,\bar{\phi}^4\Big], \end{aligned} \tag{99}$$

$$\Im V_{\text{eff}}(\bar{\phi}) = -\frac{1}{64\pi}\Big(-\mu^2 + \frac{1}{2}\lambda\,\bar{\phi}^2\Big)^2 \Theta(2\mu^2 - \lambda\,\bar{\phi}^2). \tag{100}$$

- The real part of the effective potential (99) computed with $\lambda = 0.1$ is presented in Fig. 7. In the left panel the complete effective potential is shown and in the right panel we see the one-loop contribution which is small (observe the scale) and consequently it hardly modifies the classical potential. However, it is clearly seen that the radiative correction does not tend to make the potential symmetric. It makes the minima at finite $\bar{\phi}$ even deeper. So, we conclude: the state $\bar{\phi} = 0$ is unstable and the symmetry is broken.

- The imaginary part of effective potential appears because the state $\bar{\phi} = 0$ is unstable.

- We also note that the renormalization condition (94) should be modified as

$$\frac{d^2\Re V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^2}\bigg|_{\bar{\phi}=0} = \mu^2, \tag{101}$$

  because the effective potential is complex. However, it does not change our analysis.

Massless theory

- The case of massless theory (with $m = 0$) is of special interest, as one wonders whether such a theory is symmetric or the symmetry $\phi \to -\phi$ is spontaneously broken.

- When $m = 0$, instead of Eq. (80) the effective potential equals

$$V_{\text{eff}}(\bar{\phi}) = \frac{\lambda}{4!}\,\bar{\phi}^4 + \frac{\lambda\Lambda^2}{64\pi^2}\,\bar{\phi}^2 + \frac{\lambda^2}{256\pi^2}\left[\ln\left(\frac{\lambda\,\bar{\phi}^2}{2\Lambda^2}\right) - \frac{1}{2}\right]\bar{\phi}^4 + \frac{1}{2}\delta m^2\bar{\phi}^2 + \frac{\delta\lambda}{4!}\,\bar{\phi}^4. \tag{102}$$

- It seems natural to adopt the renormalization conditions analogous to Eqs. (81) and (82) that is

$$\left.\frac{d^2V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^2}\right|_{\bar{\phi}=0} = 0, \tag{103}$$

$$\left.\frac{d^4V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^4}\right|_{\bar{\phi}=0} = \lambda. \tag{104}$$

  The first condition requires that the theory with radiative corrections included remains massless.

- Since

$$\frac{d^2V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^2} = \frac{\lambda}{2}\bar{\phi}^2 + \frac{\lambda\Lambda^2}{32\pi^2} + \frac{\lambda^2}{64\pi^2}\left[\ln\left(\frac{\lambda\,\bar{\phi}^2}{2\Lambda^2}\right) + 2\right]\bar{\phi}^2 + \delta m^2 + \frac{\delta\lambda}{2}\,\bar{\phi}^2, \tag{105}$$

  the condition (103) gives

$$\delta m^2 = -\frac{\lambda\Lambda^2}{32\pi^2}. \tag{106}$$

- Computing the fourth derivative of the potential (102), one finds

$$\frac{d^4V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^4} = \lambda + \delta\lambda + \frac{\lambda^2}{32\pi^2}\left[3\ln\left(\frac{\lambda\,\bar{\phi}^2}{2\Lambda^2}\right) + 11\right], \tag{107}$$

  which shows that the condition (104) cannot be used because $\bar{\phi} = 0$ is a singular point of the fourth derivative of $V_{\text{eff}}(\bar{\phi})$.

- So, one chooses the renormalization condition

$$\left.\frac{d^4V_{\text{eff}}(\bar{\phi})}{d\bar{\phi}^4}\right|_{\bar{\phi}=M} = \lambda, \tag{108}$$

  that is the coupling constant is determined not at $\bar{\phi} = 0$ but at $\bar{\phi} = M$. The energy parameter M is called the *renormalization scale.*

- Using the formula (107), the condition (108) provides

$$\delta\lambda = -\frac{\lambda^2}{32\pi^2}\left[3\ln\left(\frac{\lambda\,M^2}{2\Lambda^2}\right) + 11\right]. \tag{109}$$

- Substituting the counterterms (106) and (109) into Eq. (102), one obtains

$$V_{\text{eff}}(\bar{\phi}) = \frac{\lambda}{4!}\bar{\phi}^4 + \frac{\lambda^2}{256\pi^2}\left[\ln\left(\frac{\bar{\phi}^2}{M^2}\right) - \frac{25}{6}\right]\bar{\phi}^4. \tag{110}$$

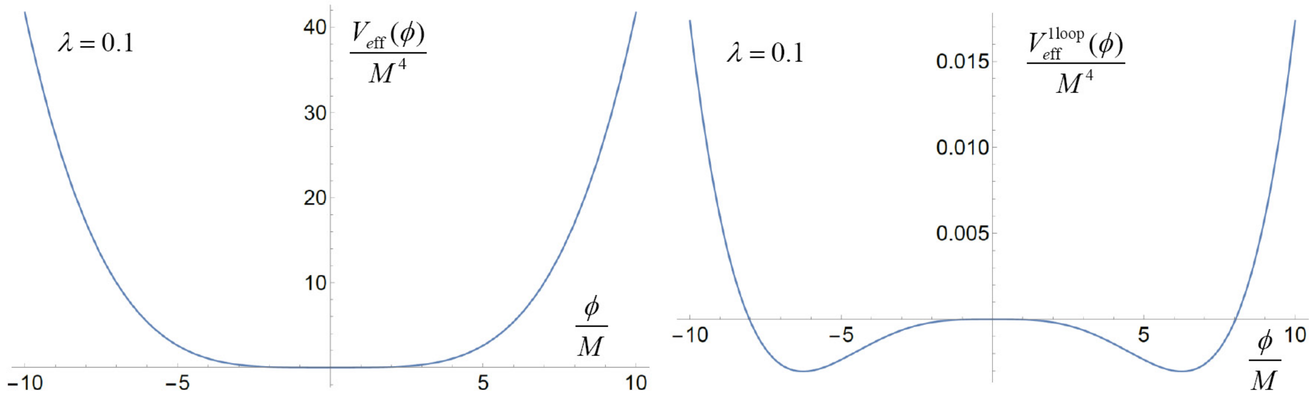

Figure 8: The effective potential (110) (left panel) and the one-loop contribution to the effective potential (110) (right panel).

- The potential computed for $\lambda = 0.1$ is shown in Fig. 8. As we see, the one-loop contribution tends to break down the symmetry. One finds that the minimum at finite $\bar{\phi}$ occurs at

$$\lambda \ln\left(\frac{\bar{\phi}^2}{M^2}\right) = -\frac{32\pi^2}{3} + \mathcal{O}(\lambda). \tag{111}$$

  Since $\lambda \ln(\bar{\phi}^2/M^2) \approx 105$, the result (111) is not reliable, as higher order contributions in the loop expansion are expected to bring higher powers of $\lambda \ln(\bar{\phi}^2/M^2)$.