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A class of continua that are not attractors of any IFS

Research Article

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Abstract: This paper presents a sufficient condition for a continuum in $\mathbb{R}^n$ to be embeddable in $\mathbb{R}^n$ in such a way that its image is not an attractor of any iterated function system. An example of a continuum in $\mathbb{R}^2$ that is not an attractor of any weak iterated function system is also given.

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1. Introduction

The notion of an iterated function system (abbrev. IFS), introduced by John Hutchinson in 1981 [4], has proven to be a fertile field of research as well as a versatile and useful tool in lossy data compression (especially where image data is concerned). This paper is a study in one specific aspect of the theory — the possibility of encoding a particular set as an attractor of an IFS. We now recall some basic terminology.

Let (X, d) be a complete metric space. A map $f\colon X \to X$ is called a *contraction* if there exists a constant $\lambda \in [0, 1)$ such that for every $x, y \in X$ we have $d(f(x), f(y)) \leq \lambda d(x, y)$. A map $f\colon X \to X$ is called *contractive* if for every $x, y \in X$, $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$, see [2]. A family $F = \{f_1, \dots, f_n\}$ of (contractive maps) contractions $f_i\colon X \to X$ is called a (*weak*) *iterated function system*, see [1]. Given a compact $B \subset X$, define

$$F(B) = \bigcup_{i=1}^{n} f_i(B).$$

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This transformation, acting on the space of nonempty compact subsets of X with the Hausdorff metric, is called the *Barnsley–Hutchinson operator*. A fixed point of F is called the *attractor* of (weak) IFS.

It is shown in [4] that every IFS has a unique attractor (an analogous fact may not be true for a weak IFS, unless compactness of X is assumed). M. Hata proved in [3] that if the attractor of some IFS is connected, then it is also locally connected. M.J. Sanders showed in [7] that every arc of finite length is an attractor for some IFS. Additionally, he has proven that if a is an endpoint of some arc $A \subset \mathbb{R}^n$ which has the properties:

(1) for all $x, y \in A \setminus \{a\}$ the length of the subarc of A with endpoints x and y is finite,

(2) for every $x \in A \setminus \{a\}$ the length of the subarc of A with endpoints x and a is infinite,

then A is not an attractor of any IFS acting on $\mathbb{R}^n$. One example of such an arc is the harmonic spiral [6]. The example of M. Kwieciński from [5] may also be easily modified to satisfy these assumptions.

It is elementary to check that every continuum in $\mathbb{R}$ is an attractor of some IFS. Moreover, any embedding of such continuum in $\mathbb{R}$ still is an attractor of some IFS. In dimension two and higher, however, the situation becomes more complex. Our results provide a sufficient condition for a continuum to be embeddable in $\mathbb{R}^n$ so that its image is not an attractor of any IFS.

2. Main results

Definition 2.1.
Let (X, d) be a metric space, $A \subset X$, $x, y \in A$, and $\varepsilon > 0$. Consider all the sequences $x_1, \ldots, x_k$ such that $k \in \mathbb{N}$, $x_1 = x$, $x_k = y$, $x_i \in A$, $d(x_i, x_{i+1}) < \varepsilon$. Denote by $\widetilde{d}(x, y, A, \varepsilon)$ the infimum of the sums $\sum_{i=1}^{k-1} d(x_i, x_{i+1})$ for these sequences. Define $\widetilde{d}(x, y, A) = \lim_{\varepsilon \searrow 0} \widetilde{d}(x, y, A, \varepsilon)$. This limit may be infinite.

It is elementary that if $A \subset B$ then $\widetilde{d}(x, y, A) \geq \widetilde{d}(x, y, B)$.

Theorem 2.2.
Let $n \geq 2$. Let $C \subset \mathbb{R}^n$ be a continuum. Assume that there exists an $(n-1)$-dimensional hyperplane $B \subset \mathbb{R}^n$ such that $B \cap C = \{p\}$ and $C \setminus \{p\}$ is connected. Assume additionally that for every $x, y \in C \setminus \{p\}$ there exists U_{xy} which is a neighbourhood of p such that $\widetilde{d}(x, y, C \setminus U_{xy}) < +\infty$. Then there exists an embedding $h\colon C \to \mathbb{R}^n$ such that $h(C)$ is not an attractor of any IFS.

Proof. By applying an affine transformation we may assume without loss of generality that $B = \{0\} \times \mathbb{R}^{n-1}$, $p = (0, \ldots, 0)$, and $C \subset [0,1] \times [-1,1]^{n-1}$. Next define $h_1, h_2\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$h_1(x_1, \ldots, x_n) = \left(x_1, \frac{x_1}{100} x_2, \ldots, \frac{x_1}{100} x_n \right), \qquad h_2(x_1, \ldots, x_n) = \left(x_1, \sqrt{x_1} \sin x_1^{-1} + x_2, x_3, \ldots, x_n \right).$$

Then define the embedding $h\colon C \to \mathbb{R}^n$ as the composition $h_2 \circ h_1$.

Roughly speaking, h_1 transforms C into a sharp needle, while h_2 bends that needle to fit into a thickened-up graph of the function $\sqrt{x} \sin x^{-1}$. As a result of the second transformation the needle becomes, speaking imprecisely, of infinite length. Figure 1 illustrates the process for $n = 2$.

The map h_1 does not increase distance, and therefore for every $x, y \in h_1(C \setminus \{p\})$ there exists U^1_{xy} which is a neighbourhood of $h_1(p)$ such that $\widetilde{d}(x, y, h_1(C) \setminus U^1_{xy}) < +\infty$. Note that, outside of any neighbourhood U of $h_1(p)$, the expansivity constant of $h_2\restriction_{h_1(C) \setminus U}$ is bounded from above. This implies that for every $x, y \in h(C \setminus \{p\})$ there exists U^2_{xy} which is a neighbourhood of $h(p)$ such that $\widetilde{d}(x, y, h(C) \setminus U^2_{xy}) < +\infty$.

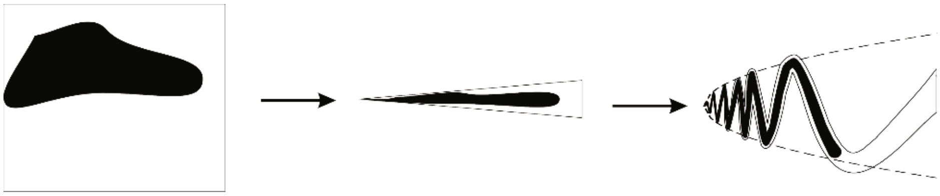

Figure 1. The map h for $n = 2$

Consider now a contraction $f: h(C) \to h(C)$ with a Lipschitz constant $\lambda < 1$. We would like to prove that if $h(p) \in f(h(C))$ then $f(h(C)) = \{h(p)\}$. To this end, conjecture that f is not constant and $h(p) \in f(h(C))$.

Assume first that $f(h(p)) = h(p)$. Fix any $x \in h(C)$ such that $f(x) \neq h(p)$. Note that the sequence $x, f(x), f^2(x), \dots$ is convergent to $h(p)$. Also note that, by the assumptions, $\widetilde{d}(x, f(x), h(C))$ is finite and additionally $\widetilde{d}(f^i(x), f^{i+1}(x), h(C)) \leq \lambda^i \widetilde{d}(x, f(x), h(C))$. But this would imply that $\widetilde{d}(x, h(p), h(C))$ is also finite, while it is not, since it can be seen from the definition of h_2 that $\widetilde{d}\left(x, h(p), h([0,1] \times [-1,1]^{n-1})\right)$ is infinite.

If, on the other hand, $f(h(p)) \neq h(p)$ then there exist $x \in h(C)$ such that $f(x) = h(p)$ and $y \in h(C) \setminus \{h(p)\}$ such that $f(y) \neq h(p)$. Then $\widetilde{d}(x, y, h(C))$ would be finite and $\widetilde{d}(f(x), f(y), h(C))$ would be infinite, which contradicts the contractiveness of f, completing the proof that if f takes value $h(p)$ on at least one argument then it has to be constant.

Consequently, if F is the Barnsley–Hutchinson operator for some IFS and $F(h(C)) \subset h(C)$, then $F(h(C))$ may comprise of $\{h(p)\}$ and possibly also finitely many other closed sets not containing $h(p)$. But then $F(h(C)) \neq h(C)$, proving that $h(C)$ is not an attractor of F. □

Remark 2.3.

The assumptions of Theorem 2.2 are technical and may seem very restrictive. Its assertion, however, is true not only for the continua that satisfy them directly, but also for the continua that are homeomorphic to subsets of $\mathbb{R}^n$ satisfying these assumptions. This significantly widens the class of sets the theorem is useful for. For example, if any two points in the continuum $A \subset \mathbb{R}^n$ can be connected in A by a path of finite length, then it can be easily seen that any one-point union of A and $[0,1]$ is homeomorphic to a subset of $\mathbb{R}^{n+1}$ for which the assumptions of Theorem 2.2 are satisfied.

After the result of Hata [3] it has been an open problem whether every locally connected continuum in $\mathbb{R}^n$ is an attractor of some IFS. The example of Kwieciński [5] provided a negative answer, but the same question for weak IFS's remains, to our best knowledge, open. We shall now give an example of a subcontinuum of $\mathbb{R}^2$ that is not an attractor of any weak IFS.

Definition 2.4.

In this definition we switch to the standard polar coordinate system (r, θ) on $\mathbb{R}^2$, that is $(x, y) = (r\cos\theta, r\sin\theta)$. Put $p_0 = (0,0)$ and $p_n = (2^{-n}, 2^{-n})$ for $n \geq 1$. For any $n \geq 1$ choose a broken line segment l_n without self-intersections, consisting of finitely many intervals, that starts at p_0, ends at p_n, has the total length of 2^n, and is contained in $\left[[0, 2^{-n}) \times (2^{-n} - 2^{-n-2}, 2^{-n} + 2^{-n-2})\right] \cup \{p_n\}$. Define $P = \bigcup_{i=1}^{\infty} l_i$.

Theorem 2.5.

The space P as a subset of $\mathbb{R}^2$ with the Euclidean metric is not an attractor of any weak IFS.

Proof. Suppose that $f: P \to P$ is contractive. We shall examine how many of the points p_i can belong to $f(P)$. If $f(p_0) \neq p_0$ then there is a neighbourhood U of $f(p_0)$ such that the distance $d(p_0, U) > 0$ and U contains finitely many points p_i and almost all of the sets $f(l_i)$. Note that only finitely many of the sets $f(l_i)$ may reach the outside of U. Also observe that each $f(l_i)$ covers at most finitely many points p_i because the lengths of l_i are not increased by f (this elementary property of contractions can be proven either by using δ-chains, or, as in [5], by using the fact that f does not increase one-dimensional measure). Consequently, only finitely many of the points p_i belong to $f(P)$.

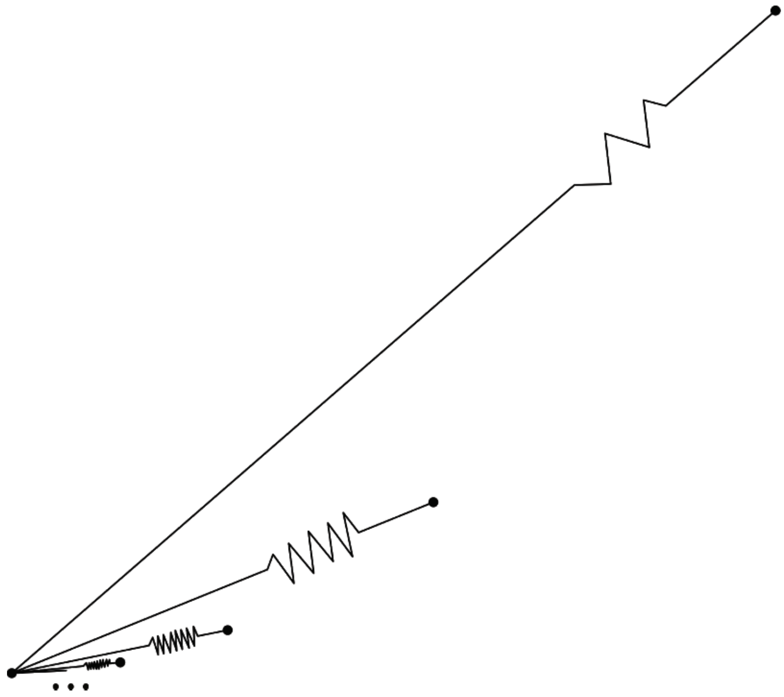

Figure 2. The space P

If, on the other hand, $f(p_0) = p_0$, then, given $n \geq 1$, note that p_n may not belong to $f(l_i)$ for $i < n$, because the lengths of these sets are too small to traverse the whole l_n. But no other point in P can be mapped onto p_n by f, because f decreases the distance between p_0 and any other point. Therefore, the only point p_i present in $f(P)$ is p_0. In conclusion, if F is a weak IFS, then only finitely many of the points p_i can belong to $F(P)$, and therefore P is not an attractor of F. □

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