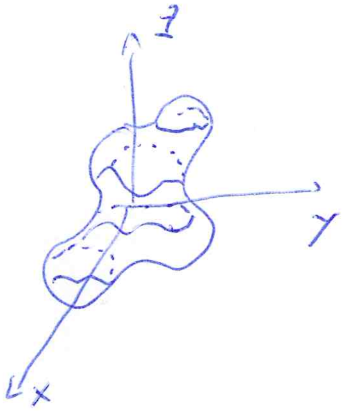
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

It means that, as we all know, the flux of $\vec{B}$ over a closed surface is zero:

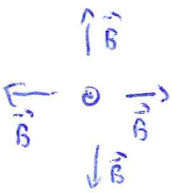
$$\phi(\vec{B}) = \int_{S_c} \vec{B}\,\vec{n}\, dS = \int_{V_{S_c}} \vec{\nabla} \cdot \vec{B}\, dV = 0$$




Obvious because there are no magnetic charges.

[Namely, if we would have a magnetic charge, then

$$\vec{B} = \frac{g}{4\pi} \frac{\vec{x}}{r^3} \qquad \int_{S_c} \vec{B}\,\vec{n}\, dS = g$$

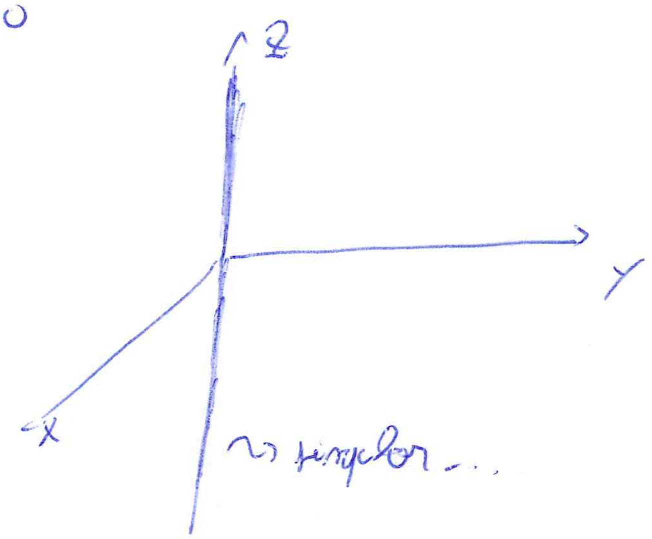


Dirac's question: can we get the field configuration of a magnetic monopole using still the laws of e.m.? And, if yes, which consequences do we have?

Let us start by studying the following potential

$$\vec{A} = \frac{g}{4\pi}\left(-\frac{y z}{r\,(x^2+y^2)}\,,\ \frac{x z}{r\,(x^2+y^2)}\,,\ 0\right)$$

singular for $x=y=0$




(So, in this form $\vec{A}$ is valid only for $r > R$, but inside another form should be valid... this is the 1st thought. But Dirac insisted in using this form also for $r \to 0$ and tried some "tricks" to make it valid).

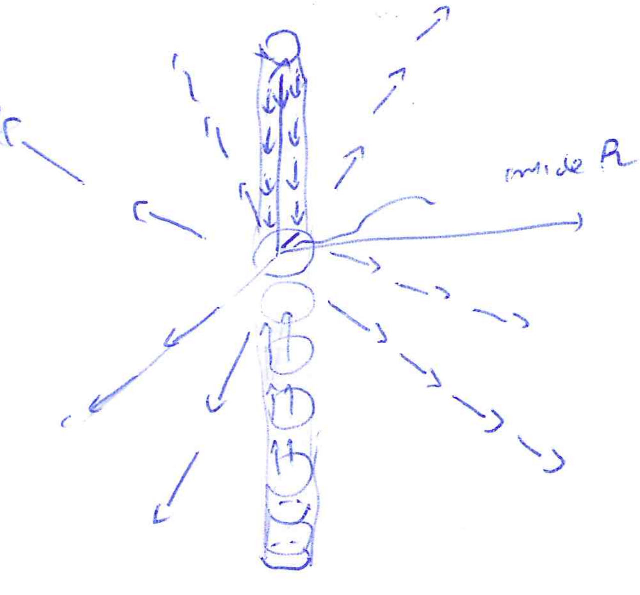
Except for $z=0$ (or a small cylinder of radius $R$), the magnetic field is:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\left.\begin{aligned} B_x &= -\partial_z A_y = \frac{q}{4\pi}\frac{x}{r^3} \\ B_y &= \partial_z A_x = \frac{q}{4\pi}\frac{y}{r^3} \end{aligned}\right\} \text{easy to calculate}$$

$$B_z = \partial_x A_y - \partial_y A_x = \frac{q}{4\pi}\frac{z}{r^3}$$ (same calculation as before, the factor $z$ is like a constant here)

$$\vec{B} = \frac{q}{4\pi}\frac{\vec{r}}{r^3}$$ looks exactly as a monopole, but $\vec{A}$ is singular...




So, if we use the full calculation with a "regular" $\vec{A}$, the total flux of $\vec{B}$ is still zero!!!

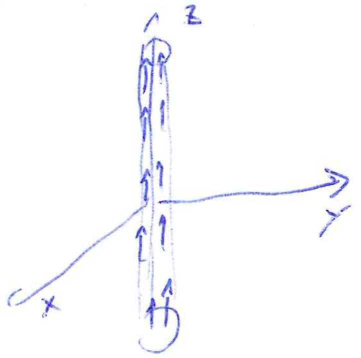
In the end, we can consider $R$ very small, but don't send it to zero!!!

Let us consider now the gauge

$$\vec{A}_0 = \frac{g}{4\pi}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

Again: singular for $z=0$, so in principle valid only for $r>R$.

$\vec{A}_0$ itself is also not physical... as we saw before in the case of Aharonov-Bohm effect, this case represent




$\vec{B} = \vec{0}$ for $r>R$.

$$\vec{A}_0 = \frac{g}{4\pi}\nabla\varphi \quad , \varphi = \arctan\left(\frac{y}{x}\right)$$

, is a pure gauge for $r>R$ (which is however singular for $r \to 0$).

Now, let us consider

$$\vec{A}_N = \vec{A} - \vec{A}_0 = \frac{Q}{4\pi}\left(\frac{z}{r}-1\right)\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

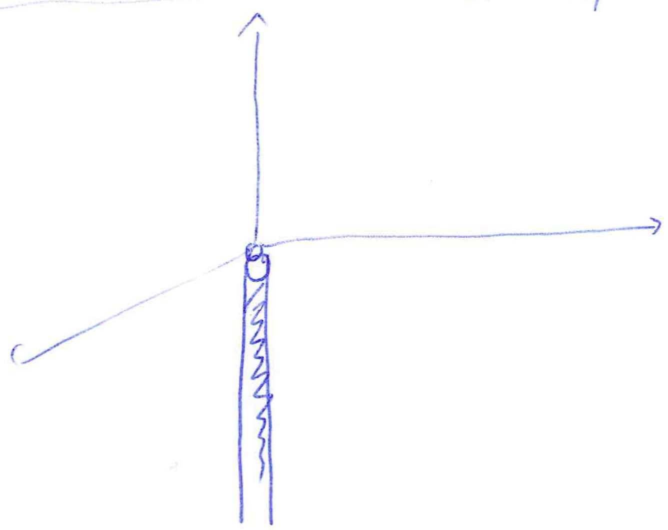
The point is that $\vec{A}_N$ is now defined for $x=y=0,\ z>0$.

Namely:

$$\boxed{\frac{z}{r}-1 = \frac{z-r}{r}}$$

is such that $=0$ for $x=y=0,\ z>0$

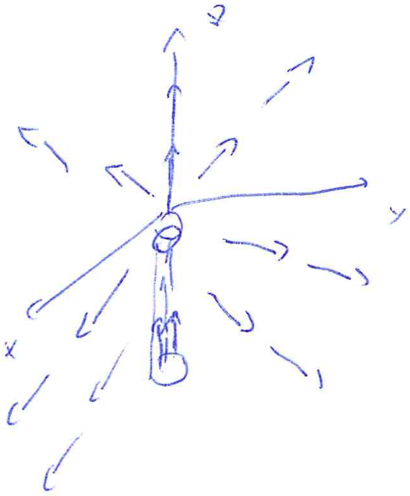
it is singular for $x=y=0,\ z\leq 0$.



$$\vec{B}_N = \frac{Q}{4\pi}\frac{\vec{r}}{r^3}$$ everywhere, except the negative $z$-axis.

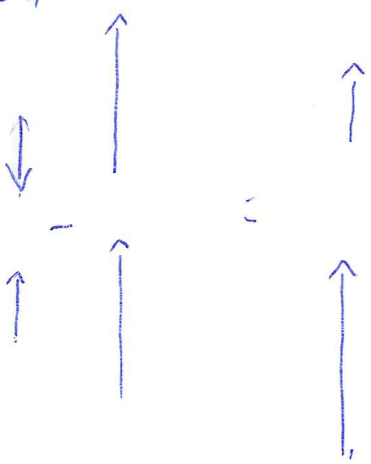
Of course, this configuration is also "unphysical", but we can imagine that as the following.

$\vec{A}_N$ corresponds to




with finite R, also if very small, we still indeed have $\phi(\vec{B}) = 0$.

Note, along the z-axis we have then (inside the coil).

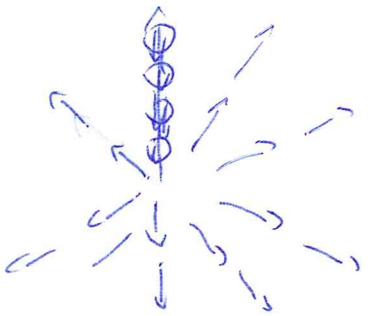


Of course, we could have directly started from $\vec{A}_N$ but the discussion would have been less intuitive.

Alternatively, we could have also started from

$$\vec{A}_S = \vec{A} + \vec{A}_0 = \frac{g}{4\pi}\left(\frac{z}{r}+1\right)\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

$$\vec{B}_S = \frac{g}{4\pi}\frac{\vec{r}}{r^3}$$ , but not on $z+$.

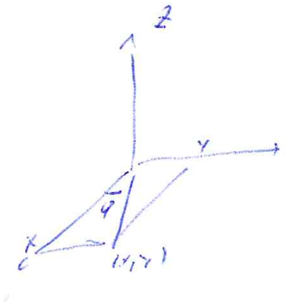


still, if I have a small circle $R \Rightarrow \phi$ still zero...

But now the main point comes; if $\vec{A}_S$ and $\vec{A}_N$ differ only because of a pure gauge, then the string is not visible.

$$\vec{A}_S - \vec{A}_N = 2\vec{A}_0 = 2\frac{g}{4\pi}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) = 2\frac{g}{4\pi}\vec{\nabla}\varphi$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

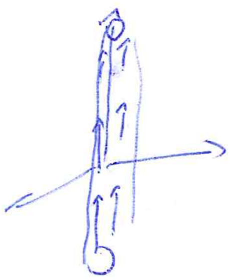



we can then write:

$$\vec{A}_S - \vec{A}_N = \vec{\nabla}\left(\frac{g}{2\pi}\varphi\right)$$

The question is now, if we can see anything like $e^{i\frac{eg}{2\pi}\varphi}$.

$$2A_0 = \frac{g}{2\pi}\left(\vec{\nabla}\varphi\right).$$



Namely, if $R \to 0$ how could we possibly see it? There is only one way to do it: via the so called Aharonov-Bohm effect.

<u>But</u> the effect of the extra phase was given by

$$\vec{A}_{gauge} = 2\vec{A}_0 = \frac{q}{2\pi}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) = \frac{q}{2\pi}\vec{\nabla}\varphi(x,y)$$

$$\underbrace{\Delta\phi = e\int \vec{A}\,d\vec{l}}_{\text{phase of the A-B effect!!!}} = \frac{eq}{2\pi}\int_0^{2\pi} d\varphi = eq = 2\pi$$

⇒ we cannot see this thin coil!!!

it is "invisible", even when using the AB effect.

More in general:

$$eq = 2\pi n$$

But then, if we set $q = \frac{2\pi}{e}$ we realise that $\vec{A}_S$ and $\vec{A}_N$ differ by a pure gauge (in the newly interpreted quantum version of it!)

That means that the "pure gauge" is

$$\vec{A}_{gauge} = \frac{1}{e}\vec{\nabla}\varphi$$

(as we could have <u>deduced</u> by inspection of the A-B effect already)

An immediate and very interesting consequence is:

quantization of charge → e is quantized.

Even one "magnetic" monopole in the universe would be enough to justify the quantization of charge, although it would be very difficult to find :)

No sense indeed: according to some theories a monopole exist in our part of the universe, which would explain why we couldn't find it.