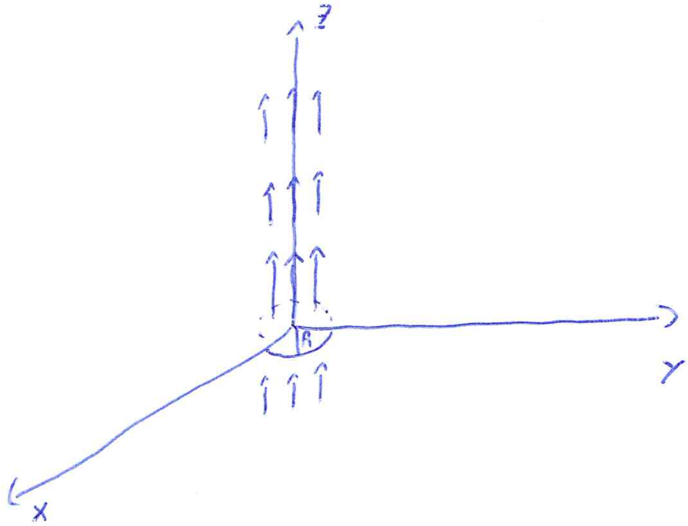
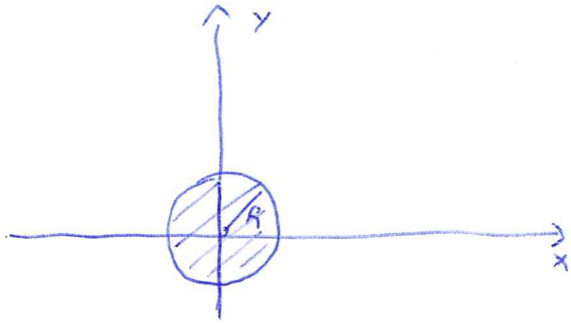
1+3 dimensions!
(finally :))





$$\vec{B} = (0, 0, B_z(x,y))$$

$$B_z(x,y) = \begin{cases} B_0 & \text{for } x^2+y^2 \leq R^2 \\ 0 & \text{for } x^2+y^2 > R^2 \end{cases}$$

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & 0 \end{vmatrix} = (0, 0, \partial_x A_y - \partial_y A_x)$$

$$\begin{cases} A_x = A_x(x,y) \\ A_y = A_y(x,y) \\ A_z = 0 \end{cases}$$

This is indeed the same solution as before, but now in the whole 3D space!

For $x^2+y^2 > R^2$

$$(A_x, A_y, A_z) = c\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

$$\partial_x A_y = \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\partial_y A_x = \frac{-(x^2+y^2) - (-y)\cdot(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\boxed{\partial_x A_y - \partial_y A_x = 0}$$

For $x^2+y^2 \leq R^2$

$$(A_x, A_y, A_z) = \frac{B_0}{2}(-y, x, 0)$$

In fact:

$$\partial_x A_y = \frac{B_0}{2} \qquad \to B_z = \partial_x A_y - \partial_y A_x = \frac{B_0}{2} - \left(-\frac{B_0}{2}\right) = B_0$$

$$\partial_y A_x = -\frac{B_0}{2}$$

The constant c is determined by requiring continuity of $\vec{A}$:

$$\boxed{\frac{c}{R^2} = \frac{B_0}{2}}$$

Digression:
is there an elegant way to obtain the solution of $\vec{A}$ without the "guessing" or with less "guessing"?

indeed, there is. To this end, let us remind the following equation (diff. eq. of 1st order)

$$\frac{df}{dx} + a(x) f = b(x)$$

$$e^{-\int a(x)dx} \frac{d}{dx}\left[f e^{\int a(x)dx} \right] = e^{-\int a dx}\left[\frac{df}{dx} e^{\int a dx} + f a e^{\int a dx} \right] = \frac{df}{dx} + fa$$

good.

$$e^{-\int a dx} \frac{d}{dx}\left[f e^{\int a(x) dx} \right] = b(x)$$

$$\frac{d}{dx}\left[f e^{\int a(x) dx} \right] = b(x) e^{\int a(x) dx}$$

$$f e^{\int a(x) dx} = \int b(x) e^{\int a(x) dx} dx + \text{const}$$

$$\boxed{f(x) = e^{-\int a(x) dx}\left[\int b(x) e^{\int a(x) dx} dx + \text{const} \right]}$$

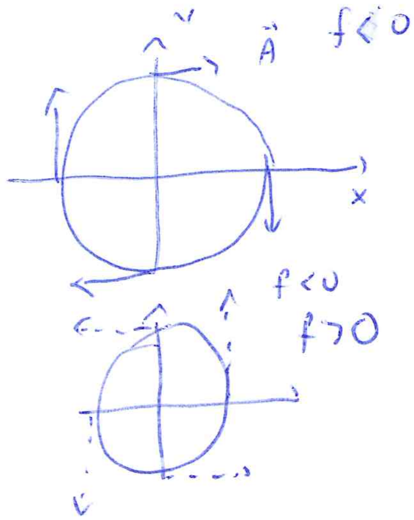
Lagrange, 1st order with arbitrary terms.

Now, let us introduce the variable

$$s = x^2 + y^2$$

Then, in a (quite) general framework one can write the following Ansatz:

$$\begin{cases} A_x = -y\, f(s) \\ A_y = x\, f(s) \\ A_z = 0 \end{cases} \qquad \begin{cases} \partial_x A_y = f(s) + x \dfrac{df}{ds} 2x \\ \partial_y A_x = -f(s) - y \dfrac{df}{ds} 2y \end{cases}$$



$$\partial_x A_y - \partial_y A_x = 2 f(s) + 2 \frac{df}{ds} \cdot s$$

Then,

$$B(x,y) \equiv B(s) = 2 f(s) + 2 \frac{df}{ds} \cdot s$$

$$\boxed{\frac{df}{ds} + f(s) \cdot \frac{1}{s} = \frac{B(s)}{2s}}$$

We can obtain each magnetic field $B(s)$.

$$f(s) = e^{-\int \frac{1}{s} ds} \left[\int \frac{B(s)}{2s} e^{\int \frac{1}{s} ds} ds + \text{const} \right]$$

$$= e^{-\ln s} \left[\int \frac{B(s)}{2s} e^{\ln s} ds + \text{const} \right]$$

$$f(s) = \frac{1}{s}\left[\int \frac{B(s)}{2}\, ds + \text{const}\right]$$

check:

a) $B(s) = 0$

$$f(s) = \frac{\text{const}}{s} = \frac{\text{const}}{x^2+y^2} \rightarrow A_x = \text{const}\cdot\frac{-y}{x^2+y^2}, \quad A_y = \text{const}\,\frac{-x}{x^2+y^2}$$

b) $B(s) = B_0$

$$f(s) = \frac{1}{s}\left[\frac{B_0}{2}\cdot s + \text{const}\right] = \frac{B_0}{2} + \underbrace{\frac{\text{const}}{s}}_{\text{set it to } 0} = \frac{B_0}{2}$$

$$\begin{cases} A_x = -y\,\frac{B_0}{2} \\ A_y = x\,\frac{B_0}{2} \end{cases}$$

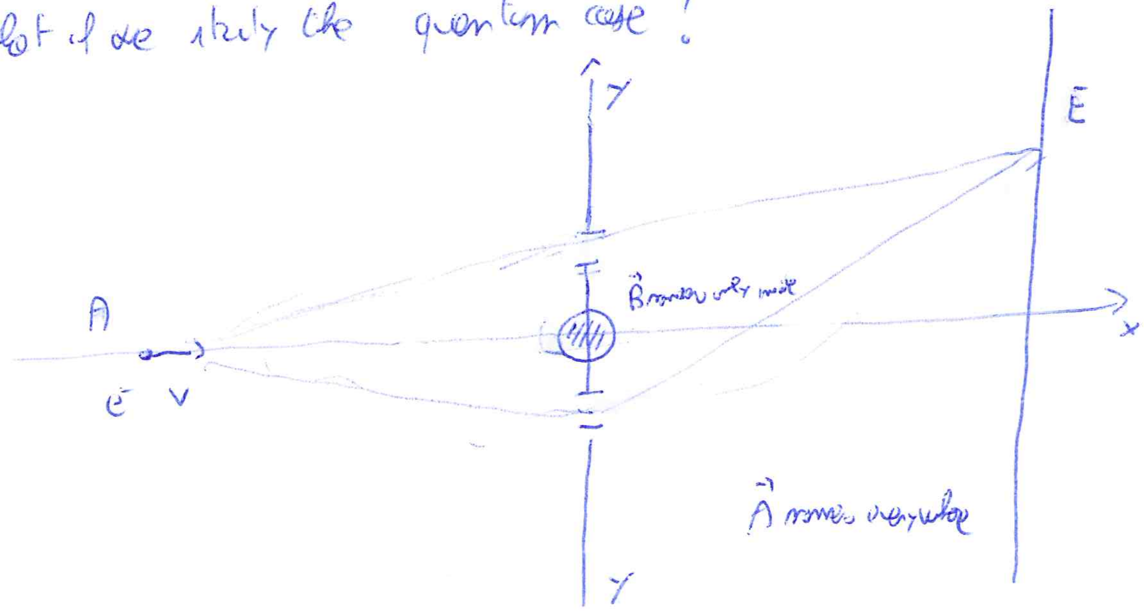
But for every form of B ~~there~~ is a choice of $f(s)$.

Why did we do it?

What is physical, $\vec{B}$ and $\vec{E}$ or $\vec{A}$?

Answer in classical field theory: $\vec{B}$ and $\vec{E}$. $\vec{A}$ is only a mathematical trick!

What if we study the quantum case?



$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2} m \dot{\vec{q}}^2 - e \vec{v} \cdot \vec{A} + \text{"double-slit"} = L_{\vec{A}=0} - e \vec{v} \cdot \vec{A}$$

Now, the probability to go from "1" to "2" in the path-integral formalism is given by:

$$\int_{\substack{\vec{q}(0) = \vec{x}_A \\ \vec{q}(T) = \vec{x}_E}} D\vec{q}\, e^{i \int dt\, L(\vec{q}, \dot{\vec{q}})}$$

Let us concentrate now on the "phase"

$$e^{i\int dt\, L(\vec{q},\dot{\vec{q}})} = e^{i\int dt\left(L_{\vec{A}=\vec{0}}(q,\dot{q}) - e\vec{v}\vec{A}\right)} = e^{i\int dt\, L_{\vec{A}=\vec{0}}} \cdot e^{-ie\int dt\, \vec{v}\cdot\vec{A}}$$

$$= e^{iS_{\vec{A}=\vec{0}}} \cdot e^{-ie\int d\vec{x}\cdot\vec{A}}$$

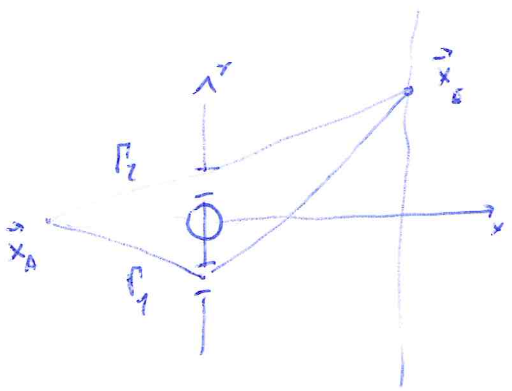
Now, let us consider the two main contributions, which we call

$\vec{x}_1(t)$

and

$\vec{x}_2(t)$.

$$\int\limits_{\substack{\vec{x}(0)=\vec{x}_A \\ \vec{x}(T)=\vec{x}_E}} D\vec{x}\, e^{iS} \approx \underbrace{\#}_{\text{small fluctuations}}\left(e^{iS(\vec{x}_1(t))} + e^{iS(\vec{x}_2(t))}\right) \doteq \langle\vec{x}_E|e^{-iHT}|\vec{x}_A\rangle$$



$$\begin{cases} S(\vec{x}_1(t)) = S^{(\vec{x}_1)}_{\vec{A}=\vec{0}} - ie\int_{\Gamma_1} d\vec{x}\cdot\vec{A} \\ S(\vec{x}_2(t)) = S^{(\vec{x}_2)}_{\vec{A}=0} - ie\int_{\Gamma_2} d\vec{x}\,\vec{A} \end{cases}$$

Then, we get:

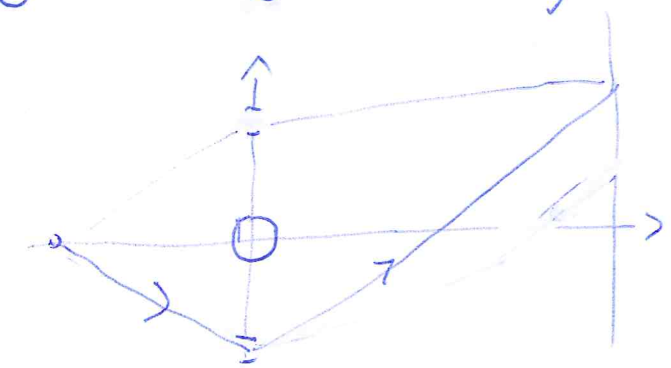
$$\text{Path integral} = \langle \vec{x}_E | e^{-iHT} | \vec{x}_A \rangle =$$

$$= e^{iS_1} + e^{iS_2} = e^{iS_{1,\vec{A}=\vec{0}}} \cdot e^{-ie\int_{\Gamma_1} d\vec{x}\,\vec{A}} + e^{iS_{2,\vec{A}=\vec{0}}} e^{-ie\int_{\Gamma_2} d\vec{x}\cdot\vec{A}} =$$

$$= e^{iS_{1,\vec{A}=\vec{0}}} e^{-i\int_{\Gamma_1} d\vec{x}\cdot\vec{A}} \cdot \left(1 + e^{i\left(S_{2,\vec{A}=\vec{0}} - S_{1,\vec{A}=\vec{0}}\right)} \cdot e^{-ie\int_{\Gamma_2} d\vec{x}\cdot\vec{A} + ie\int_{\Gamma_1} d\vec{x}\cdot\vec{A}}\right)$$

$$\left(e^A + e^B = e^A\left(1 + e^{B-A}\right)\right)$$

$$= e^{iS_1}\left(1 + e^{i\Delta\phi(\vec{A}=\vec{0})} \cdot e^{+ie\oint d\vec{x}\cdot\vec{A}}\right)$$



Now, $\displaystyle\oint_{\Gamma_{tot}} d\vec{x}\cdot\vec{A} = \int_{A(\Gamma_{tot})} \vec{B}\cdot\vec{n}\cdot dS = \phi(\vec{B}) = B_0 \pi R^2 \neq 0$

Namely $\displaystyle = \int_{A(\Gamma_{tot})} \vec{B}\,\vec{n}\,dS = \int_{A(C)} \vec{B}\,\vec{n}\,dS$

Ergo,

$$\Delta\phi_{tot} = \Delta\phi(\vec{A}=\vec{0}) + e\cdot B_0 \pi R^2 = \Delta\phi(\vec{A}=\vec{0}) + \text{const}$$

↳ we can change this constant by changing the radius R and/or B_0.

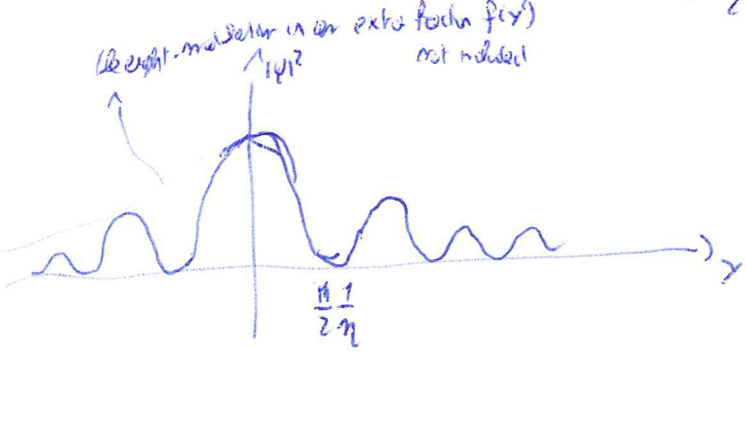
$\Delta\phi(\vec{A}=\vec{0})$ is a function of y on the screen... it usually takes the value

$$\Delta\phi(\vec{A}=\vec{0}) = \eta y$$

$\vec{A}=\vec{0}$

$$|1+e^{i\eta y}|^2 = 1+1+2\cos(\eta y) = 2(1+\cos(\eta y))$$

$y = \frac{\pi}{2}\cdot\frac{1}{\eta}$

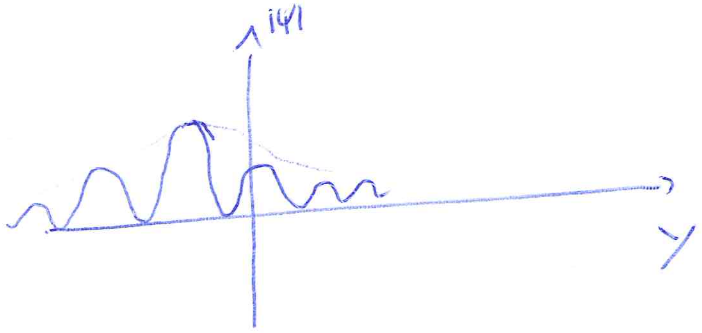


Interference fringes.

Now, with $\vec{B} \neq \vec{0}$, and therefore with $\vec{A} = \vec{0}$, we get:

$$\mapsto \left|1+e^{i(\eta y + c_0)}\right|^2 = 2\left(1+\cos(\eta y + c_0)\right)$$

$c_0 > 0$



Then, the presence of the magnetic field influences the position of the fringes.

Note that:

- the paths Γ_1 and Γ_2 do not go through C_R
- paths which go through C_R give a negligible contribution to the wave function.

→ It follows that $\vec{A}$ is the physical object, because it is indeed even more fundamental than $\vec{B}$....

It is in fact $\vec{A}$ which is coupled to the charge-particle, and not $\vec{B}$ which is away... ↳ $\vec{A}$ is "locally coupled to it".

Wonderful "Gedankenexperiment" which unifies e.m. and QM.

Later, it has been realized and the theoretical prediction exper.
seen.

"one of the seven wonders of the quantum world"