VORTEX in 1+2 dims

1+2 dimension

Start point: $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}(\vec{\phi}^2 - F^2)^2$$

O(2) symmetry

$\vec{\phi} \mapsto B\vec{\phi}$

Introducing the complex field one can obtain an equivalent formulation:

$\phi = \phi_1 + i\phi_2$

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - \frac{\lambda}{2}(\phi^*\phi - F^2)^2$$

U(1) symmetry

$\phi \mapsto e^{i\varphi}\phi$

There is a conserved current:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^*$$

(Noether current)

$\delta\phi = i\alpha\phi$

$\delta\phi^* = -i\alpha\phi^*$

$$J^\mu = i\alpha(\partial^\mu\phi^* \cdot \phi - \partial^\mu\phi \cdot \phi^*)$$

If $\phi \equiv \phi(x,y)$ space like only, then we are left only

$$J^i = -i\alpha(\partial_i\phi^*\phi - \partial_i\phi\,\phi^*)$$

Consideration: I have said before that O(2) does not admit solitons....
then, what are we looking for? Indeed, for an interesting solution which needs an "extra element" to be realised.

Let us consider the e.o.m.

$$[\partial_x^2+\partial_y^2]\phi_1 = \lambda\phi_1(\phi_1^2+\phi_2^2-F^2)$$

with ϕ_1, ϕ_2

$$[\partial_x^2+\partial_y^2]\phi_2 = \lambda\phi_2(\phi_1^2+\phi_2^2-F^2)$$

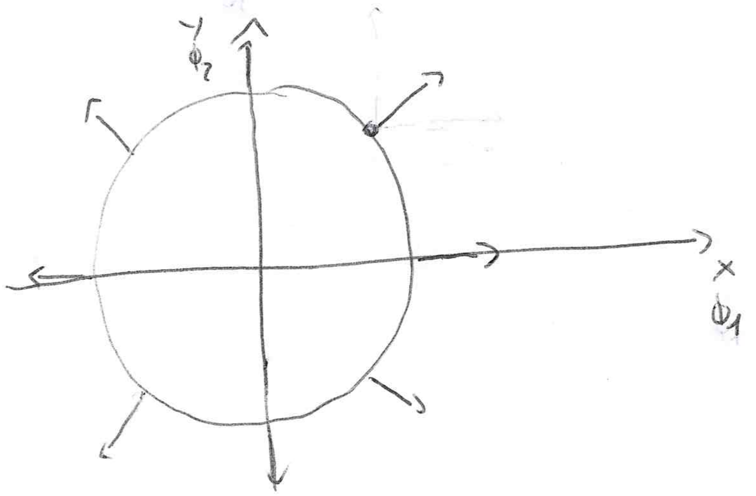
Using $\phi = \phi_1 + i\phi_2$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \to \Box\phi^* = -\frac{\lambda}{2}\cdot 2\phi^*(\phi^*\phi-F^2) \hookrightarrow \Delta\phi^* = \lambda\phi^*(\phi^*\phi-F^2)$$

$$\Delta\phi = \lambda\phi(\phi^*\phi-F^2)$$

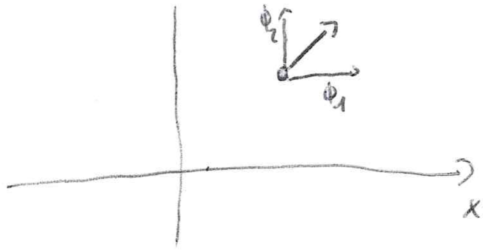
Let us consider:

$$(\phi_1,\phi_2) = \left(\frac{Fx}{\sqrt{x^2+y^2}}, \frac{Fy}{\sqrt{x^2+y^2}}\right) \quad \boxed{\phi_1^2+\phi_2^2 = F^2} \to \phi = Fe^{i\arctan\left(\frac{y}{x}\right)}$$



But "Achtung"... ϕ_1, ϕ_2 are internal (isospin) d.o.f., while (x, y) is the external space.

Note: singular for $x, y \to 0$. So, indeed that configuration



For each (x,y) I can attach a $\vec{\phi} = (\phi_1, \phi_2)$ pole.

In order to visualize it I'd need 4 dimensions...

Is this a solution of the eom?

$$\Delta\phi_1 = \lambda\phi_1(\phi_1^2+\phi_2^2-F^2) = 0$$

$$\Delta\phi_1 = ?$$

$$\phi_1 = F\frac{x}{\sqrt{x^2+y^2}}$$

$$\partial_x\phi_1 = F\cdot\frac{\sqrt{x^2+y^2} - x\cdot\frac{x}{\sqrt{}}}{x^2+y^2} = F\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\partial_y\phi_1 = -\frac{Fxy}{(x^2+y^2)^{3/2}}$$

$$\partial_x^2\phi_1 = -Fy^2\cdot 2x\cdot\frac{3}{2}(x^2+y^2)^{-5/2}$$

$$\partial_y^2\phi_1 = -Fx\cdot\frac{(x^2+y^2)^{3/2} - y\,2y\,\frac{3}{2}(x^2+y^2)^{1/2}}{(x^2+y^2)^3} = \frac{-Fx}{(x^2+y^2)^{3/2}} + \frac{3y^2x}{(x^2+y^2)^{5/2}}$$

$$= -\frac{Fx(x^2-2y^2)}{(x^2+y^2)^{5/2}} \rightarrow (\partial_x^2+\partial_y^2)\phi_1 \neq 0.$$

So, this is not a solution of the eom.

Note however that:

$$\partial_x^2 \phi_1 + \partial_y^2 \phi_1 = -\frac{F x}{(x^2+y^2)^{3/2}} \quad \longmapsto 0 \text{ as } 1/r^2 \quad !!!$$

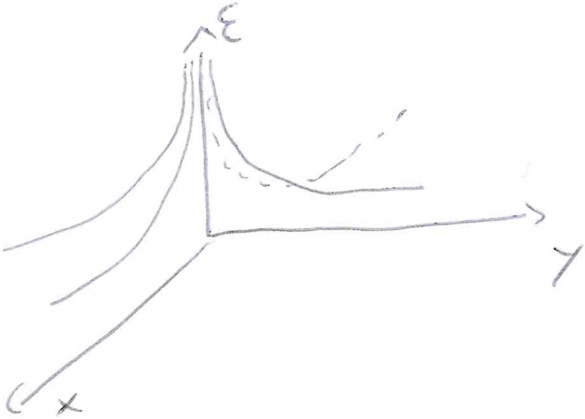
So the leading term cancels...

=

Energy of this configuration:

$$\mathcal{E} = \frac{1}{2}(\partial_x \phi_1)^2 + \frac{1}{2}(\partial_y \phi_1)^2 + \frac{1}{2}(\partial_x \phi_2)^2 + \frac{1}{2}(\partial_y \phi_2)^2$$

$$\mathcal{E} = F^2 \left[\frac{y^4}{(x^2+y^2)^3} + \frac{2x^2y^2}{(x^2+y^2)^3} + \frac{x^4}{(x^2+y^2)^3} \right] = \frac{F^2}{x^2+y^2}$$



How is the total energy of this configuration?

$$E = \int dx\,dy\, \mathcal{E} = 2\pi \int_0^\infty r\,dr\, \frac{F^2}{r^2} = 2\pi F^2 \int_0^\infty dr\, \frac{1}{r}$$

There are two ∞...

We can namely see that

$$E = 2\pi F^2 \int_{\varepsilon}^{\Lambda} dr \, \frac{1}{r} = 2\pi F^2 \ln \frac{\Lambda}{\varepsilon} \longmapsto \infty \ldots$$

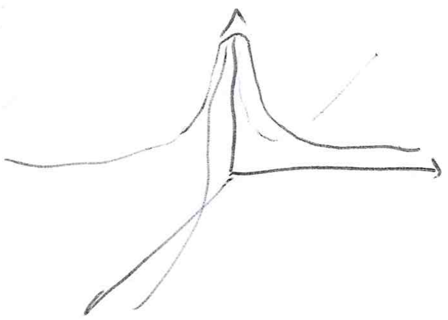
but ... a real configuration $\vec{\Phi}(x,y)$ should not have singularity.

what we are considering is only an asymptotic behaviour.

we study $\vec{\phi}(x,y)$ which for $x^2+y^2 > R^2$ behaves as $\left(\frac{Fx}{\sqrt{x^2+y^2}}, \frac{Fy}{\sqrt{x^2+y^2}} \right)$.

$$E = 2\pi \int_0^{\infty} dr \, \mathcal{E}(r) = \underbrace{2\pi \int_0^{R} dr \, \mathcal{E}(r)}_{\text{something finite}} + \underbrace{2\pi \int_R^{\infty} dr \, \mathcal{E}(r)}$$

↳ this is unfortunately still ∞ !!!



2π

So, 3 questions

* it is not a sol of the eom
* E is divergent, even if we cure the $x=y=0$ singularity
* where is the vortex? It looks as an hedgehog!

We first answer the third question.

The conserved current $\vec{J}$ takes the form:

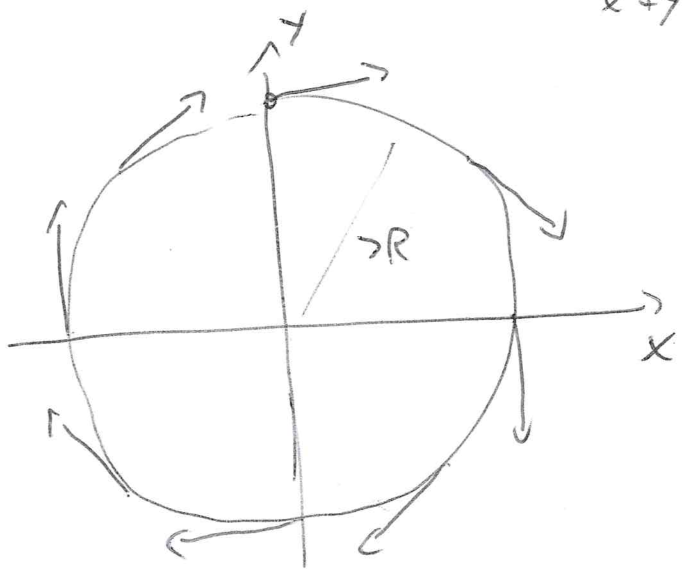
$$\vec{J} = F^2 \left(\frac{y}{x^2+y^2} , \frac{-x}{x^2+y^2} \right)$$

Namely:

$$J_\mu = (\partial_\mu \phi_1)\phi_2 - (\partial_\mu \phi_2)\phi_1$$

$$J_1 = \partial_x \phi_1 \cdot \phi_2 - \partial_x \phi_2 \cdot \phi_1 = F^2 \frac{y}{x^2+y^2}$$

$$J_2 = \partial_y \phi_1 \phi_2 - \partial_y \phi_2 \phi_1 = -F^2 \frac{x}{x^2+y^2}$$



Introduction of the Electromagnetic field in the O(2) model

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$$

$\boxed{\mu,\nu = 0,1,2}$

(1+2 dims)

$$\mathcal{H} = \mathcal{E} = -F^{0k} F^{0}_{\ k} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

For a space-like field $A_\mu(x,y)$ we have:

$$\mathcal{E}(x) = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{4} F^{kl} F_{kl}$$

$\boxed{k,l = 1,2}$

$\left[D = 1 \rightarrow \mathcal{E}(x) = 0 \right]$

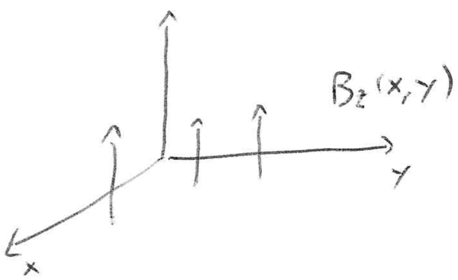
$D = 2$

$$\mathcal{E} = \frac{1}{2}\left(F_{12}\right)^2 . \qquad F_{12} = \partial_1 A_2 - \partial_2 A_1$$

$D = 3 \rightarrow$

$$\mathcal{E} = \frac{1}{2} \vec{B}^2$$

We can think to the $D=2$ case as a plane $(D=2)$ embedded in a 3-d world where only B_z is $\neq 0$.



$$A_3 = 0$$

$$A_1 = A_1(x,y)$$

$$A_2 = A_2(x,y)$$

If an insect is living on the plane, he will see that there is a Lorentz force acting on charged particles.

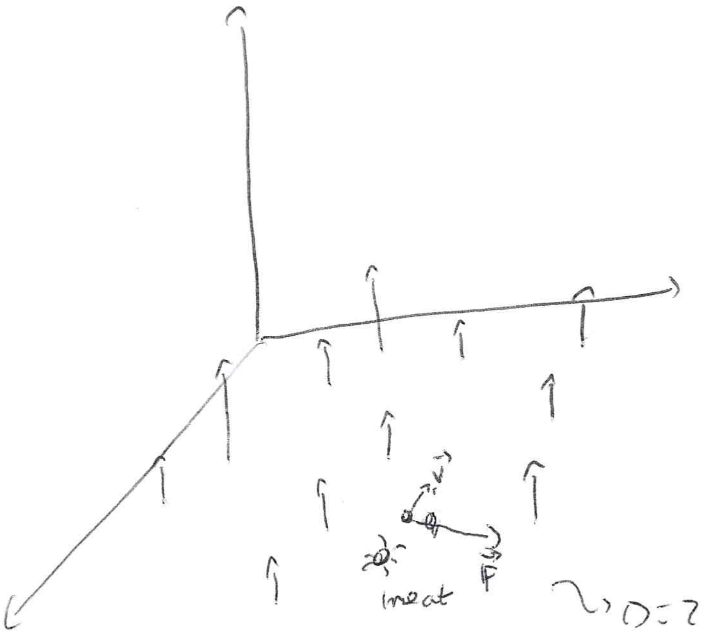
$$\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & 0 \end{vmatrix} \qquad \begin{array}{l} A_x = A_x(x,y) \\ A_y = A_y(x,y) \end{array}$$

$$= (0, 0, B_z = \partial_x A_y - \partial_y A_x)$$

$$\vec{F} = q\,\vec{v} \wedge \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_x & v_y & 0 \\ 0 & 0 & B_z \end{vmatrix} = \vec{i}\,(q B_z v_y) - \vec{j}\,(q v_x B_z).$$

$$F_x = q v_y B_z$$

$$F_y = -q v_x B$$



$\vec{A}_k(x,y)$

$k = 1, 2$

(Note: the third dimension is for our visualization... it is not indeed needed. Everything would be consistent in the two-dim world without imagining it).

Let us put together the e.m. field and the U(1) model.

$$\mathcal{L} = [D_\mu \phi]^* [D^\mu \phi] - \frac{\lambda}{2} (\phi^* \phi - F^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} . \quad D_\mu = \partial_\mu - ie A_\mu$$

$$\phi \mapsto e^{-i\alpha(x)} \phi$$

$$A_\mu \mapsto A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

$$D_\mu \phi \mapsto e^{-i\alpha(x)} D_\mu \phi \qquad x = (t, x, y)$$

Now, let us start again with

$$\vec{\phi} = (\phi_1, \phi_2) = F \left(\frac{x}{\sqrt{x^2+y^2}} , \frac{y}{\sqrt{x^2+y^2}} \right) \quad \text{for } r > R$$

what does it mean for ϕ complex?

$$\phi = F e^{i (\arctan \frac{y}{x})} = F e^{i\varphi(x,y)} \qquad \varphi(x,y) = \arctan\left(\frac{y}{x}\right)$$

$$D_i \phi = \partial_i \phi - ie A_i \phi = F(i \partial_i \varphi) e^{i\varphi} - ie A_i F e^{i\varphi}$$

That is, we realize that if

$$A_i = \frac{1}{e} \partial_i \varphi \rightarrow \boxed{D_i \phi = 0} \quad \text{for } r > R .$$

$\mathcal{E}(x) = 0$ for $r > R$. The e.m. field "eats" up the part coming from ϕ.

$[D_\mu \phi]^* [D^\mu \phi]$

$$\vec{A} = (A_x, A_y) = \frac{1}{e}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Note that:

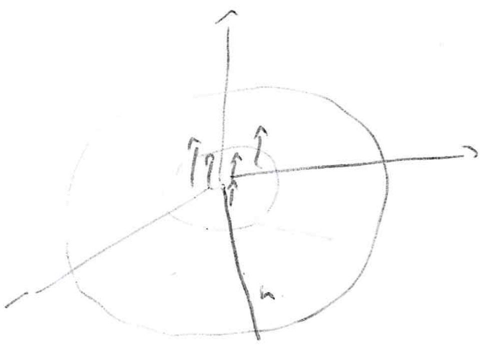
$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = 0 \qquad \left(\text{obviously, } A_i = \frac{1}{e}\partial_i \varphi \text{ is a gauge}\right).$$

But $\vec{A}$ is also singular for $x, y \mapsto 0$.

This form is valid only if understood as a form valid for $r > R$.

For $r < R$ what do we have?

There must be a magnetic field inside.



$$\boxed{\phi = \int_{C_r, r>R} B_z \, dx\, dy = \int_{C_r} \nabla \times \vec{B} \cdot d\vec{l} = \int_{C_r} \vec{A} \cdot d\vec{l} = \frac{2\pi}{e}}$$

Aah! We have a "magnetic monopole" with magnetic charge $\frac{2\pi}{e}$.

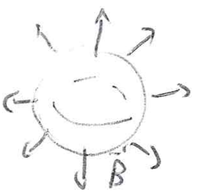
We have indeed "vortex + monopole". Finite energy.

$\phi \qquad A_\mu$

$\vec{B}$

Important points:

- $\vec{B}$ is zero for $r > R$, $\vec{A}$ is not. $\int \vec{A}\, d\vec{l} \neq 0$ means that a "regular" $\vec{A}$ is such that we have in the center $\vec{B} \neq \vec{0}$.

- The concept of "magnetic monople" is vague here...  ... $\Phi(\vec{B})$

In order to see it better we will study a model in 3 spacial dimensions.

For the moment it is just a name...

we have in the 2dim world a magnetic structure.

Next we...