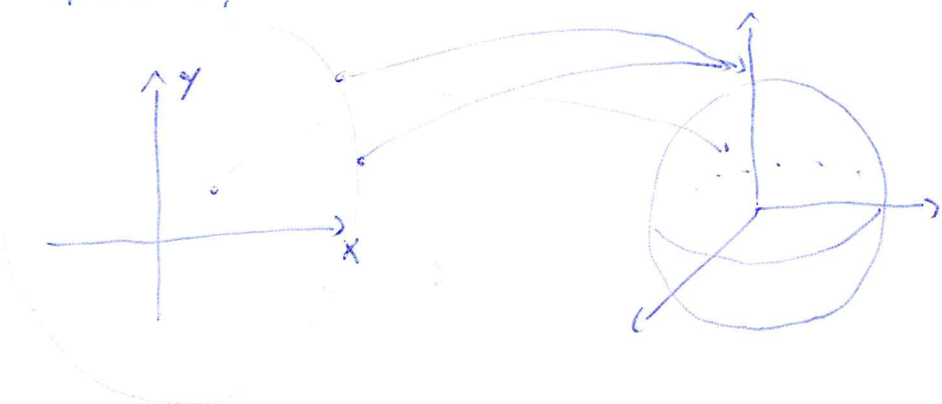
1+2 dimensions. O(3) model, Relax.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}\left(\vec{\phi}^2 - F^2\right)^2 \qquad \vec{\phi} = \vec{\phi}(t,x,y) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$\boxed{\text{Invariant under: } \vec{\phi} \mapsto B\vec{\phi}, \; B \in SO(3)}$

We consider only $\vec{\phi}(x,y)$ (space-like)

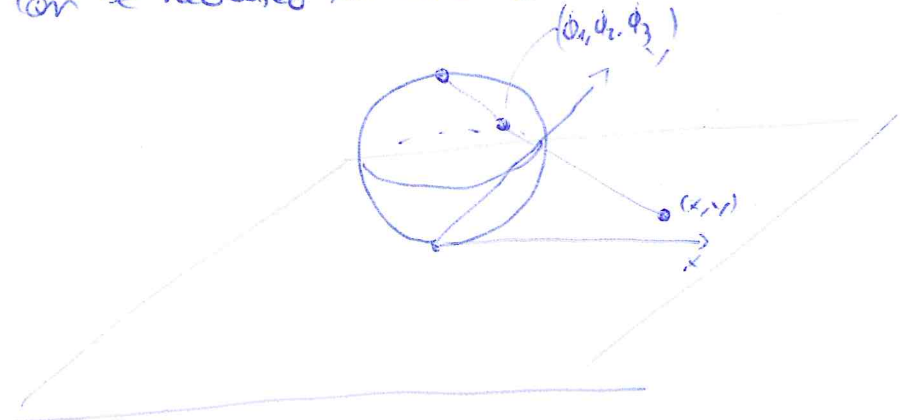
$$\vec{\phi}(r,\varphi) \; ; \; \vec{\phi}(r \to \infty, \varphi) = \vec{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \forall \varphi .$$



$$\boxed{\phi : S_2 \longmapsto S_2}$$

This mapping is non-trivial.... $[S_2 \mapsto S_1$ on the contrary would be trivial.$]$

It can be realised for instance via the ster. coordinates



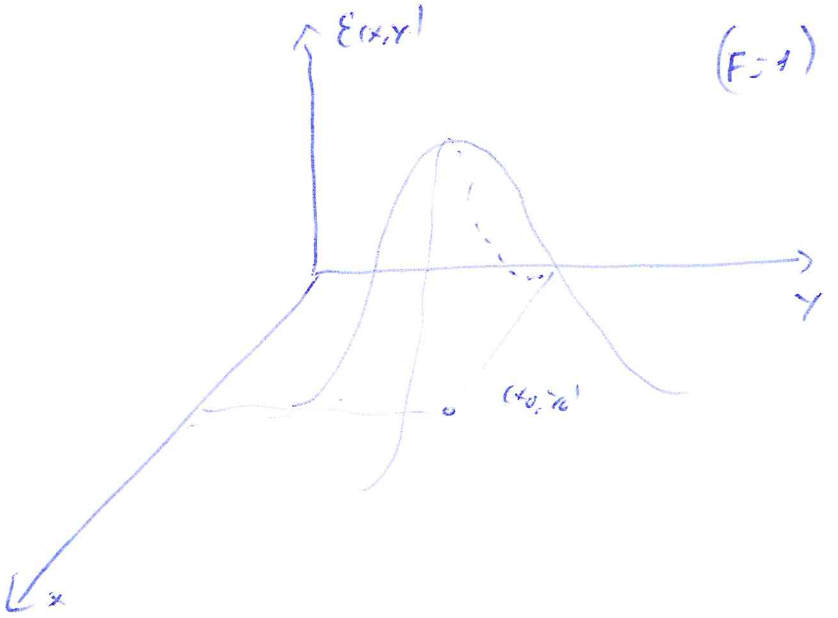
$$Q = \frac{1}{8\pi}\int d^2x \, \varepsilon_{\mu\nu} \vec{\phi}\left(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}\right) \qquad \mu,\nu = 1,2 \equiv x,y$$

$\in n$

$\vec{\phi}_{sol}(x,y)$ with $n=1$ are such that

$$\mathcal{E}(x,y) = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^2 = \left[\frac{\lambda\ \lambda_0}{\lambda_0^2 + \frac{(x-x_0)^2+(y-y_0)^2}{4}}\right]^2$$

[Namely, for $D=2$ it must be $V_2 = 0$ always, $\frac{\lambda}{4}\left(\vec{\phi}^2 - F^2\right)^2 = 0$, $\vec{\phi}^2(x,y) = F^2$
$\forall\, x,y$]



lump with dimension $1/\lambda_0$.

The important point is that λ_0 is not fixed... each dimension is ok. The overall energy

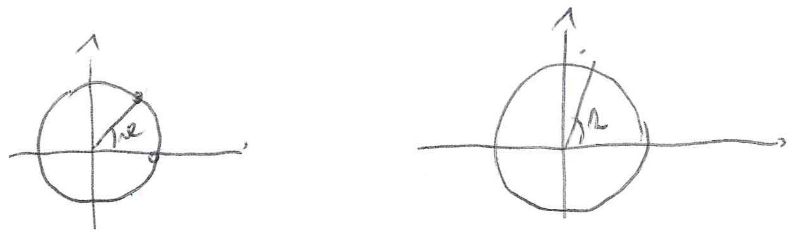
$$E = 4\pi$$

is indep. on λ.

This is a "solitonic solution".

Digression: $S_1 \to S_1$

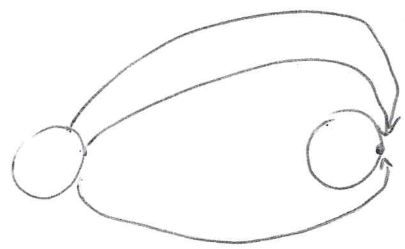
Classification of non-singular mappings ...



$\vartheta \mapsto \Lambda$

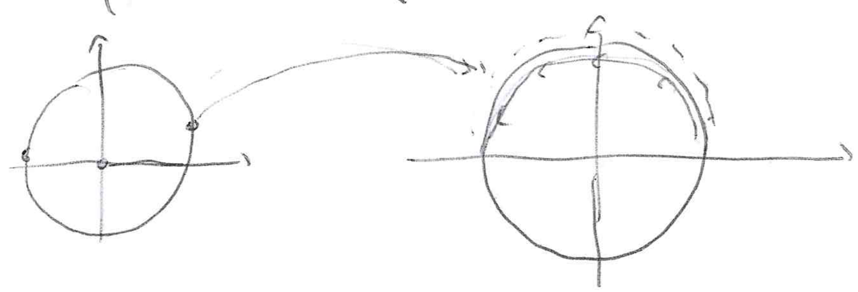
$\Lambda(\vartheta)$. $\Lambda(0) = \Lambda(2\pi)$

$\Lambda(\vartheta) = 0 \to$

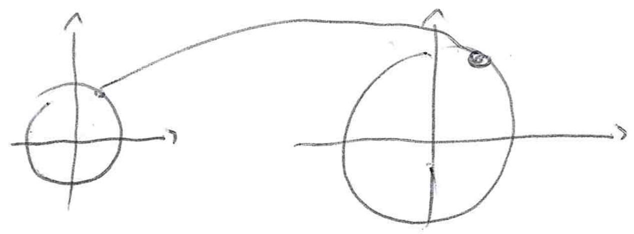


trivial

$$\Lambda(\vartheta) = \begin{cases} t\,\vartheta & 0 \le \vartheta < \pi \\ t\,(2\pi - \vartheta) & \pi \le \vartheta < 2\pi \end{cases}$$



$\Lambda(\vartheta) = \vartheta$



$$Q = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Lambda}{d\vartheta}\, d\vartheta$$

$\Lambda(\vartheta) = n\,\vartheta$ 1 round here $\mapsto$ n round there ... Q is the homotopy class ...

$\Lambda(0) = \Lambda(2\pi)$

One winds:

$$\Pi_1(S_1) = \mathbb{Z}.$$

Indeed, for $S_2 \mapsto S_2$ it is similar, although more complicated...

$$\Pi_2(S_2) = \mathbb{Z}$$

$$Q = \frac{1}{8\pi}\int d^2x \; \varepsilon_{\mu\nu} \, \vec{\phi} \, (\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})$$

Classifies these homotopy classes.

$Q \in \mathbb{N}$.

Soliton in 1+2 $\longmapsto$ Instantons in 1+1.

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}(\vec{\phi}^2 - F^2)^2}$$

$$\rightarrow \boxed{\mathcal{L}_E = \frac{1}{2}(\partial_\mu^E \vec{\phi})^2 + \frac{\lambda}{4}(\vec{\phi}^2 - F^2)^2} \qquad \mu = 1, 2 \qquad \partial_2 = \partial_Y$$

$$Z = \int D\phi \, e^{-\overbrace{\int dx\, dY \, \mathcal{L}_E}^{S_E}}$$

We then can study the Thermodynamics by

$$Z(T) = \int_{\substack{PBC \\ \phi(0,x) = \phi(\beta,x)}} D\phi \, e^{-\int_{-\infty}^{\infty} dx \int_0^{\beta = 1/T} dY \, \mathcal{L}_E}$$

$\vec{\phi}_{sol}(x,y) = F(x,y)$ is the soliton in the 1+2 model $\quad \mathcal{E}(x,y)$

$\vec{\phi}_{inst}(x,Y) = F(x,Y)$ is the "instanton" in the 1+1 model $\quad \mathcal{L}(x,Y)$

$\hookrightarrow$ "pseudoparticle"

Minimum of the action

Interesting:

one can also construct $\vec{\phi}_{col}(x,Y)$ such that $\vec{\phi}_{col}(x,0) = \vec{\phi}_{col}(x,\beta)$ and $Q = n$

$\rightarrow$ important role at nonzero T

Heuristic expansion in the topological sectors

$$Z = \sum_{n} \int_{\substack{BC \\ Q=n}} D\phi \, e^{-\mathcal{L}_E}$$

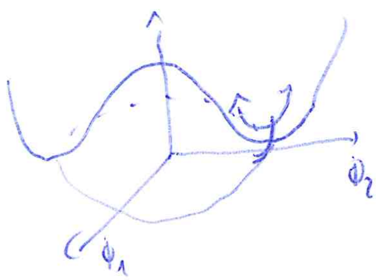
"Division in topological classes"

—

Mermin - Coleman - Wagner theorem:

Naively, you would say: "I pick up a vacuum.... then I have one massive fields and two massless (Goldstone bosons) fields.

But here it is not the case; indeed, we realize that $\vec{\phi}_{inst}(x,\tau)$ is exactly responsible for "mixing" all the classical vacua.



Nonlinear limit:

$\lambda \mapsto \infty$, F is kept konstant. $\vec{\phi}^2 = F^2 \quad \forall x, \tau$

$$Z = \int_{BC} D\phi \, e^{-\int dx d\tau \left[\frac{1}{2}(\partial_\mu^E \vec{\phi})^2 + \frac{\lambda}{4}(\vec{\phi}^2 - F^2)^2\right]} =$$

$$= \int_{BC} D\phi \, \delta(\vec{\phi}^2 - F^2) \, e^{-\int dx d\tau \frac{1}{2}(\partial_\mu^E \vec{\phi})^2}$$

$[\phi] = [E^0]$ in 1+1.

F is also dimensionless...

ergo, in this model there is no dimensionful parameters. Just as a YM theory.

3 d.o.f $\to$ 1 constraint... then you would naively expect two particles, that is 2 d.o.f.

Moreover, the naive expectation is to have two massless fields with $m = \infty$

$\hookrightarrow$ but it is in the end very different!

Properties:

dimensionless theory (conformal invariance). Moreover: renormalizable.

In the vacuum it turns that you don't have spontaneous symm. breaking.

$\langle \vec{\phi} \rangle = 0$.

But each $\vec{\phi}$ is such that $\vec{\phi}^2 = F^2$, this is the constraint! $\rightsquigarrow$ $\vec{\phi}_{int}$ is such that $\langle \vec{\phi}_{int} \rangle = 0$ and

All 3 fields ϕ_1, ϕ_2, ϕ_3 get a mass. No breaking of symmetry, therefore they get the same mass.

$$m^2 = \Lambda^2 e^{-4\pi/g_0^2}$$

↳ bare coupling, very small

Λ = high energy scale, very large

This is at the same time:

- dim. transmutation (breaking of the trace anomaly due to quantum fluctuations)
- mass gap; in the beg you have massless particles, which get a mass due to quantum fluctuations.

m is a finite number. Note: we can't calculate it!

$g_0 \mapsto g(\mu)$

$$\mu \frac{d g^2}{d\mu} = -\frac{g^2}{2\pi} < 0$$

↳ the model shows also asymptotic freedom, just as a YM theory.

Thus, O(3) in 1+1 does not describe pions but rather → gluons!!!

→ see talk

Finite temperature:

- at low T: 3 dof
- at high T: 2 dof

very peculiar properties

(similar to a single gluon ...)

At high T everything is "perturbative" $\rightarrow$ 2 dof.

- $M \propto T$ for high T $\rightarrow$ it must!