Recall : Derrick's theorem

1+D dims.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - V \quad ; \quad \vec{\phi} = \vec{\phi}(\vec{x}) \text{ space-like only}$$

$$E = \int d^D x \left[\frac{1}{2} \partial_\mu \vec{\phi})^2 + V \right] = V_1 + V_2$$

$$\begin{cases} V_1 = \int d^D x \, \frac{1}{2} \partial_\mu \vec{\phi})^2 \geq 0 \\ V_2 = \int d^D x \, V \geq 0 \end{cases}$$

For a soliton solution $\vec{\phi}_{sol}(\vec{x})$ it must be

$$\boxed{(2-D) V_1 = D V_2}$$

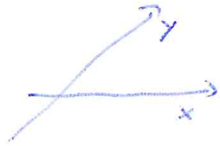
$D \leq 2$

Proven by dilatation, $\vec{\phi}_\lambda(\vec{x}) = \vec{\phi}_{sol}(\lambda \vec{x}) \ldots$ E_λ must have a minimum for $\lambda = 1$.

No solutions for $D = 2, 3, \ldots$

We have studied up to now $D=1$. "Line-worlds" in the Flatland language...

let us go to the 'real' flatland, that is $D=2$. (Edwin Abbott: Flatland)

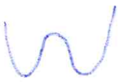


We study here the case in which

$$\vec{\phi} = (\phi_1, \phi_2, \phi_3)$$

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}(\vec{\phi}^2 - F^2)^2}$$

This is the so-called $O(3)$ model in $1+2$ dimension. (Why $O(3)$? $\vec{\phi} \mapsto R\vec{\phi}$, $R \in O(3)$, is an invariant)



- Linear form: λ is finite

- Non-linear form. $\lambda \mapsto \infty$ (F is kept constant)

$\Downarrow$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2$$

but there is in addition the constraint $\vec{\phi}^2 = F^2$ (confined on a circle)!

- ACHTUNG: No spont. breaking of symmetry

Why "3" and not "2" and not "4"?

This can be clarified by group operations, see later on.

Derrick's tells us:

- $D=2 \to V_2 = 0$.

$\vec{\phi}_{sol}$ must be such that $\vec{\phi}_{sol}^{\,2} = F^2$!

In this sense, for the soliton there is no difference between linear and non-linear vs of the model.

$$E = \int d^2x \; \frac{1}{2}(\vec{\nabla}\vec{\phi})^2 = \int d^2x \; \frac{1}{2}\left[(\partial_x\vec{\phi})^2 + (\partial_y\vec{\phi})^2\right]$$

A finite-energy solution must be such that, for each $\vec{x} \mapsto \infty$ (no matter the direction!) it must tend to a "unique value"

$$\boxed{\vec{\phi}(\vec{x} \mapsto \infty) = \vec{\phi}_0}$$

$$\boxed{\phi_i(r \mapsto \infty, \theta) = \phi_{i,0} \quad \forall \theta!}$$

In fact: in polar coordinates, it is possible to show that

$E \mapsto \infty$, if $\vec{\phi}$ is dependent from θ!

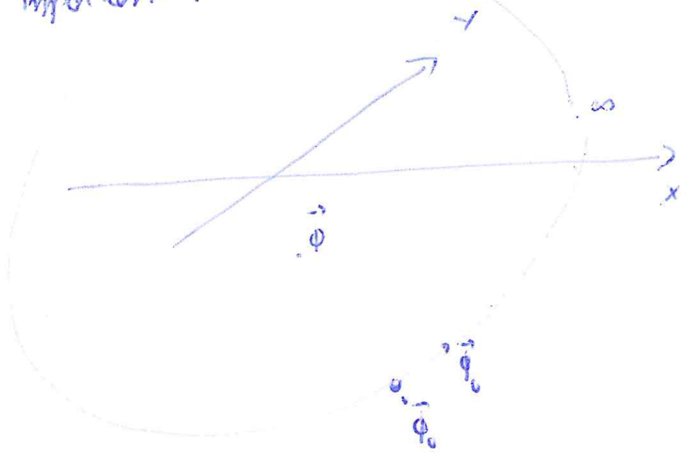
Namely:

$$(\vec\nabla\phi_i)^2 = \left(\frac{\partial\phi_i}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_i}{\partial\theta}\right)^2$$

$$\int d^2x\,(\vec\nabla\phi_i)^2 \quad \text{contains the term} \quad \int r\,dr\,d\theta\,\frac{1}{r^2}\left(\frac{\partial\phi_i}{\partial\theta}\right)^2 = \int_0^\infty \frac{dr}{r}\underbrace{\int_0^\infty d\theta\left(\frac{\partial\phi_i}{\partial\theta}\right)^2}$$

if this is not zero for $r \to \infty$ you get a divergent result.

This is important:



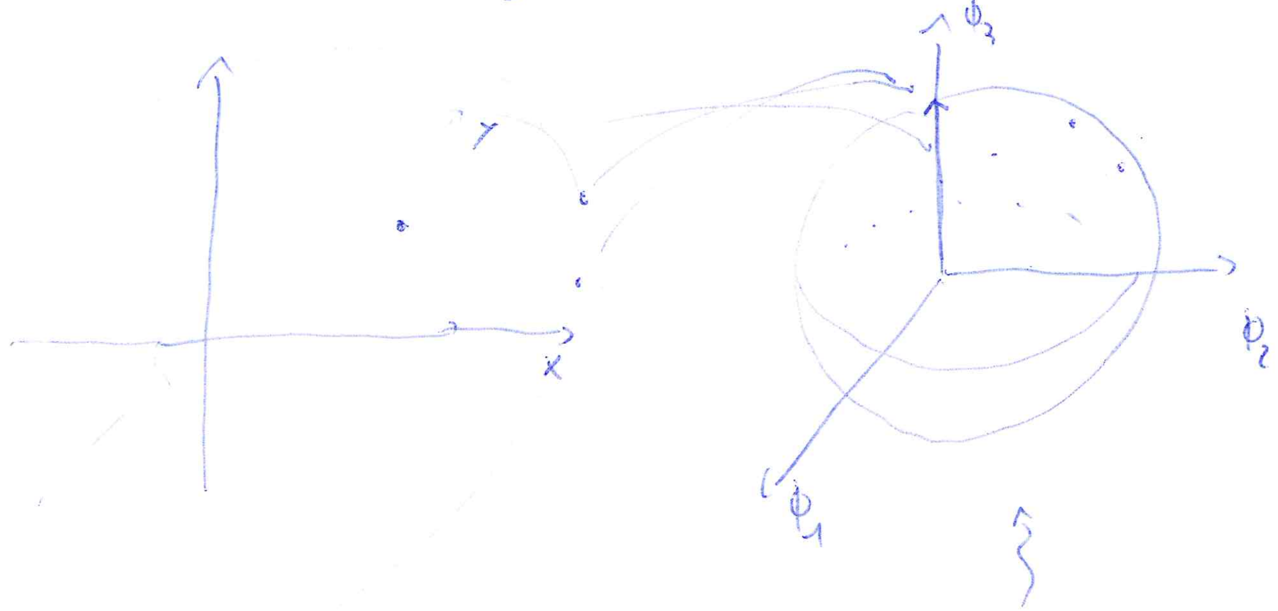
There is only one point at spatial ∞..

Note:

- this is very $\neq$ from the one dimensional case... in a sense, it is opposite to that!!! There: different values for $x \to \infty$ and $x \to -\infty$ were necessary...

The point is that the structure of the infinity is completely different in the two cases.. on a line: two distinct values, on a plane: a connected circle!!!

Then, we have the following situation:



it is not a circle

$\vec{\phi}_{sol}(x,y)$ · from the plane to the sphere ...

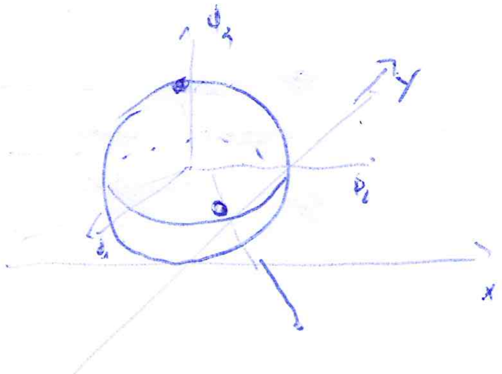
but due to the extra feature at spatial ∞ we have a nontrivial situation:

"internal space" : "external space"

sphere

x-y-plane with condition on infinity (that is: also a sphere!!!)

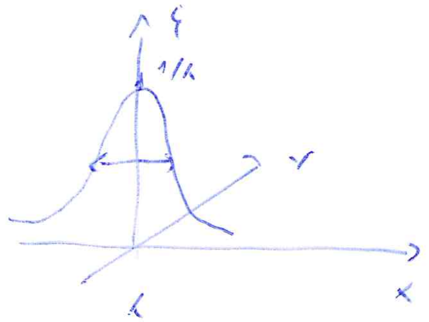
For instance, the mapping can be the stereographic construction



Mathematical expressions somewhat lengthy, but the idea is clear...

Using this stereographic projection you get ($F=1$)

$$\begin{cases} \dfrac{2\phi_1}{1-\phi_3} = \dfrac{\lambda x}{x^2+y^2} \\[2ex] \dfrac{2\phi_2}{1-\phi_3} = -\dfrac{\lambda y}{x^2+y^2} \end{cases}$$



$$\phi_3 = \sqrt{1-\phi_1^2-\phi_2^2}$$

$$E = \int d^2x \left[\frac{\lambda}{\lambda^2+\frac{x^2+y^2}{4}}\right]^2$$

Translation

$$\left(E = \int d^2x \left[\frac{\lambda}{\lambda^2+\frac{(x-x_0)^2+(y-y_0)^2}{4}}\right]^2 \right)$$

Boosts

Important:

λ is not restricted ... each value of λ is ok ... there are solutions of each size. In fact:

$$E = 2\pi\int_0^\infty r\,dr \left[\frac{\lambda}{\lambda^2+\frac{r^2}{4}}\right]^2 = 4\pi\,!$$ (indep. on λ ...)

this is a property of Q(1) as well...

Note, by reintroducing F we have.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}\left(\vec{\phi}^2 - F^2\right)^2$$

$$[\mathcal{L}] = E^3$$

$$[\vec{\phi}] = E^{1/2}$$

$$E = 4\pi F^2 = E_{sol} \; !!!$$

More in general, one can construct solitons with the energy

$$\boxed{E = 4\pi F^2 \cdot n}$$

it looks like a conserved "charge"...

$$Q = \frac{1}{8\pi}\int d^2x \; \varepsilon_{\mu\nu} \; \vec{\phi}\left(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}\right) \qquad \mu, \nu = 1, 2 = x, y \; .$$

$$\boxed{E = 4\pi |Q|}$$

Note also: the existence of these solitons is the reason why we do not have a spontaneous breaking of chiral symmetry!!!

Why for "3" and not for "2" and "4"?

For 3, we have a mapping of a sphere in a sphere:

$$S_2 \longleftrightarrow S_2$$

The mapping is not trivial!!!

For ϕ_1, ϕ_2 you have a mapping

$$S_2 \longleftrightarrow S_1$$

which is trivial.

(Example of the rubber band...).

For $\phi_1, \phi_2, \phi_3, \phi_4$

$$S_2 \longmapsto S_3$$

also trivial... thus, the only possibility is indeed ϕ_1, ϕ_2, ϕ_3!!!

D = 1, O(3) model in the non linear case

D = 1

1+1, but quantum version

$$\begin{cases} \mathcal{L} = \frac{1}{2} (\partial_\mu \vec{\phi})^2 & [\phi] = E^0 \\ |\vec{\phi}|^2 = F^2 \mapsto \text{dimensionless!!!} \end{cases}$$

$$Z = \int_{PBC} D\phi \, e^{-S_E} = \int_{PBC} D\phi \, e^{-S}$$

Now, for x, Y there are "instantons", with each dimensions!!!

Overlapping instantons when studying multisolution!

ACHTUNG: No spontaneous breaking of symmetry!!!

⟹ See talk in KOS

How many d.o.f.?

At low T: 3 with mass (effective mass)

At high T: 2 with no mass (perturbative result)

$\boxed{\text{Lattice: } N = 3!!!}$