Integrals and tunneling and instantons

Recall that:

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}}$$

Basic integral in (quantum) field theory.

Recall also:

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 + Jx} = \sqrt{\frac{2\pi}{a}}\, e^{J^2/2a}$$

Generalizations:

N-case

$$\int dx_1 \dots dx_N\, e^{-\frac{1}{2}x^t A x + Jx} = \sqrt{\frac{(2\pi)^N}{\det A}}\, e^{\frac{1}{2}JA^{-1}J} \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}. \qquad A = (N \times N)$$

QFT (functional case, $N \mapsto \infty$):

$$\int D\varphi\, e^{-\frac{1}{2}\varphi \cdot K \cdot \varphi + J\varphi} \doteq e^{\frac{1}{2} J \cdot K^{-1} \cdot J}$$

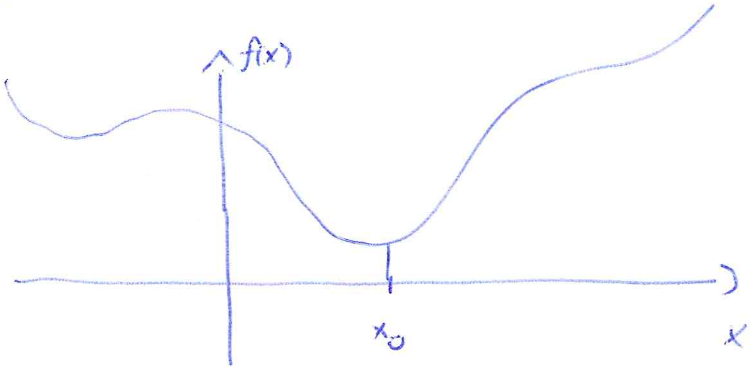
$\varphi(x)$ is seen as an infinite vector, actually with a countless infinity

see QFT-II course

Now, suppose that you want to calculate the integral

$$I = \int_{-\infty}^{\infty} e^{-f(x)} \, dx$$

where $f(x)$ is some function limited from below.



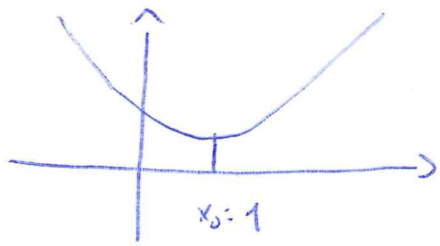
Suppose that the global minimum is given for $x = x_0$, then we can make a Taylor expansion:

$$f(x) = f(x_0) + \underbrace{f'(x_0)}_{=0}(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \dots$$

$$I \simeq \int_{-\infty}^{\infty} e^{-f(x_0) - \frac{1}{2} f''(x_0)(x-x_0)^2} dx = e^{-f(x_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} f''(x_0)(x-x_0)^2} dx = \underbrace{e^{-f(x_0)}}_{\text{exponential term}} \underbrace{\sqrt{\frac{2\pi}{f''(x_0)}}}_{\text{small fluctuation}}$$

Example:

$f(x) = \cosh(x-1)$



$f(x_0 = 1) = 1$

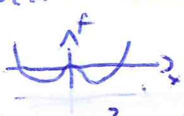
$f(x) = 1 + \frac{1}{2}(x-1)^2$

$$\int_{-\infty}^{\infty} e^{-\cosh(x-1)}\,dx \quad e^{-1}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2}\,dx = e^{-1}\cdot\sqrt{2\pi} \simeq 0.92$$

The exact integral is 0.84 ... ✓

=

$f(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ → this is trickier... because there are two minima:



$f(1) = -1/4 = f(-1)$

$f'(x) = -x + x^3 = 0 \rightarrow x = \pm 1 \rightarrow f''(x) = -1 + 3x^2 \mapsto f''(1) = 2$

$$\int_{-\infty}^{\infty} e^{-f(x)}\,dx = \int_{-\infty}^{\infty} e^{+1/4}\cdot e^{-\frac{1}{2}\cdot 2\cdot(x-1)^2}\,dx + \int_{-\infty}^{\infty} e^{1/4}\, e^{-\frac{1}{2}\cdot 2\,(x+1)^2}\,dx \simeq$$

$$\simeq 2\cdot e^{1/4}\cdot\sqrt{\frac{2\pi}{2}} \simeq 4.5$$

The exact result is 3.9 ✓ ... but the factor 2 is important...

By the way:

$$I = \int_{-\infty}^{\infty} e^{-f(x)}\, dx$$

is a QFT in 0+0 dimensions... In fact, if you consider a QFT in $\overbrace{1+D-1}^{D \text{ total dim}}$ dim you have

$$Z_E = \int D\varphi\, e^{-\int d^D x\, \mathcal{L}_E} \qquad \mathcal{L}_E = \frac{1}{2}(\partial_\mu^E \varphi)^2 + V(\varphi)$$

for $D=0$ this is a simple integral

$$Z_E = \int_{-\infty}^{\infty} d\varphi\, e^{-V(\varphi)}$$

So, this QFT is a simple number...

This example is

⇝ Asymptotic series and QFT

⇝ S. Keitel's work and all approximations

⇝ Extension to Finite Temperature

Now, let us go to the quantum system described by $\boxed{t = -i\tau}$

$$L = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \mapsto \boxed{L_E = \frac{1}{2}\left(\frac{d\varphi}{d\tau}\right)^2 + V(\varphi)}$$

"Euclidean Lagrangian"

$L(t=-i\tau) = -L_E$

$$\langle \varphi_1 | e^{-HT} | \varphi_2 \rangle = \int_{\substack{\varphi(-T/2) = \varphi_1 \\ \varphi(T/2) = \varphi_2}} D\varphi \, e^{-S_E} = \sum_n \langle \varphi_1 | n \rangle \langle n | \varphi_2 \rangle e^{-E_n T}$$

$$S_E = \int_{-T/2}^{T/2} d\tau \, L_E$$

if you calculate this you can get the energy level...

it is in the same form as our integral I ...

S_E is a functional of φ ...

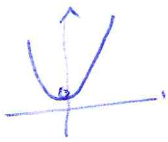
The first thing to do is to find the function $\varphi(\tau)$, which fulfills the boundary conditions and minimizes the euclidean action S_E.

$$\delta S_E = 0$$

$$\boxed{\ddot{\varphi}(\tau) = \frac{\partial L_E}{\partial \varphi} = \frac{\partial V}{\partial \varphi}}$$

euclidean e.o.m.

of course, you may have "trivial solutions"...

$V(\varphi) = \frac{m^2}{2}\varphi^2 + \dots$  $\langle 0| e^{-HT} |0\rangle$

$\varphi(\tau) = 0 = \text{const} \to S_E = 0$.

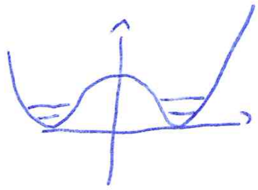
Then, the fluctuations will deliver the usual E_n...

but there are some cases, where more minima are present, in which

some nontrivial $(\varphi(\tau) \neq \text{const})$ may and do exist...

let us go back to the usual Double-Well-potential...

For $V(\varphi) = \frac{\lambda}{4}(\varphi^2 - F^2)^2$

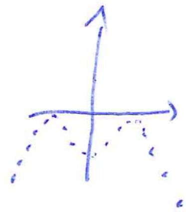


we have seen that

$$\underset{(n=1)}{E_A} - \underset{(n=0)}{E_S} \sim \langle +F | e^{-HT} | -F \rangle \sim e^{-S_E^{MIN}} \quad (T \to \infty)$$

where S_E^{MIN} is such that: $\varphi(-T/2) = -F$, $\varphi(T/2) = +F$ and minimizes: $\int_{-T/2}^{T/2} dY\, L_E$.

That is, it fulfills the boundary condition and is a solution of $\ddot{\varphi} = \frac{\partial V}{\partial \varphi}$



$$\varphi_{inst}(Y) = F \tanh\left[\frac{m}{2}(Y - Y_0)\right], \quad S_E = S_E^{MIN} = S_E^{inst} = M_{KINK}$$

(just as the soliton-kink...)

Then,

$$E_A - E_S \propto e^{-S_E^{inst}}$$

The instanton is a euclidean solution of the e.o.m. which is non-trivial (it is not a constant) and has finite action.

it appears typically when studying the path integral in QFT, $\int D\varphi\, e^{-S_E}$.

(Also at nonzero temperature, where $Z = \int_{PBC} D\varphi\, e^{-S_E}$)

Note:
(The fluctuations are then typically very complicated... $\# e^{-S_E^{inst}}$
↳ very hard to calculate)

COMPARISON of GENERAL DEFINITIONS

Soliton: $\phi(t,\vec{x})$

Instanton: $\varphi(x_E)$ $\quad x_\mu = \begin{pmatrix} t \\ x_1 \\ \vdots \\ x_{D-1} \end{pmatrix} \mapsto \vec{x}_E = ($

$1+D$ dimensions = $D+1$ dim.

$1+(D-1)$ dimensions = D dims.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi)$$

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu^E\varphi)^2 + V(\varphi) = \frac{1}{2}(\vec{\nabla}_E\varphi)^2 + V(\varphi)$$

$\phi(\vec{x}) \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_D \end{pmatrix}$

$$(\partial_t^2 - \Delta)\phi(\vec{x}) = -\frac{\partial V}{\partial\phi}$$

$$\boxed{\Delta\phi(\vec{x}) = \frac{\partial V}{\partial\phi}}$$

$\varphi(\vec{x}_E) \quad \vec{x}_E = \begin{pmatrix} x_1 \\ \vdots \\ x_{D-1} \\ x_D \equiv Y \end{pmatrix}$

$$\boxed{\Delta_E\phi(\vec{x}_E) = \frac{\partial V}{\partial\phi}}$$

Soliton: finite energy (space-like) nontrivial solution of the e.o.m.

$$E = \int d^Dx \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right]$$

nontrivial: $\phi(\vec{x}) \neq$ const, $E > 0$.

Instanton: finite action nontrivial solution of the euclidean e.o.m.

$$S_E = \int d^Dx_E \left[\frac{1}{2}(\vec{\nabla}_E\phi)^2 + V(\phi)\right]$$

$\phi(\vec{x}_E) \neq$ const, $S_E > 0$.

Meaning: static wave (stable, no spreading) (by boosting you can get moving waves).

Classical object

Meaning: "tunnelling" contribution to a path integral of the the type $\int_{BC} D\varphi\, e^{-S_E}$ in which the boundary conditions allow for it.

Quantum object.. $e^{-S_E^{inst}}$ is a contributi