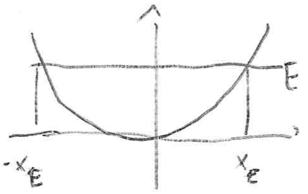
Field theory in 1+0 dim.

Classical mechanics (recall):

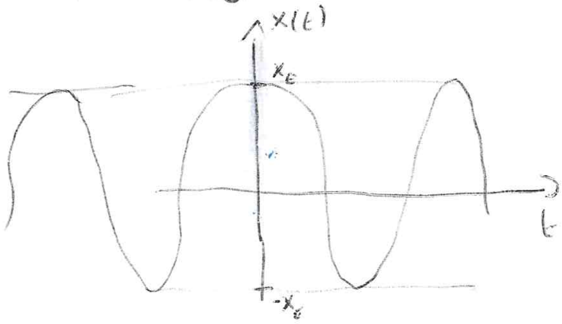
$$L = \frac{1}{2} m_0 \dot{x}^2 - V(x) \xrightarrow{\text{eom}} \boxed{m_0 \ddot{x} = -\frac{dV}{dx}}$$

This is a classical field theory in 1+0 dimensions.

[Achtung ... you may say, it is two dimensional, such as 1+1 ... $\uparrow^t \to_x$ ... but even if this is true, one direction depends on the other. Having a trajectory, you indeed have $x = x(t)$ ... So it is a classical field theory with one variable.]

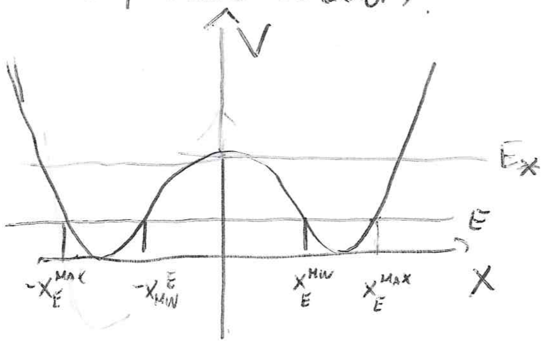


Each E is allowed ...

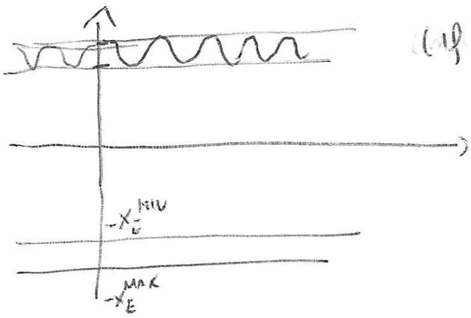


($V(x) = \frac{1}{2} m_0 \omega^2 x^2 \mapsto x(t) = x_0 \cos(\omega t)$, otherwise one has a more complicated function, but still some periodic behavior).

$$V(x) = \frac{\lambda}{4}(x^2 - F^2)^2$$



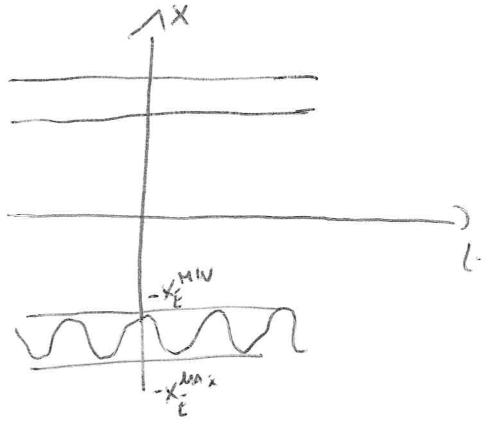
$E < E_*$



(if $x_0 \in (x_E^{MIN}, x_E^{MAX})$ ...

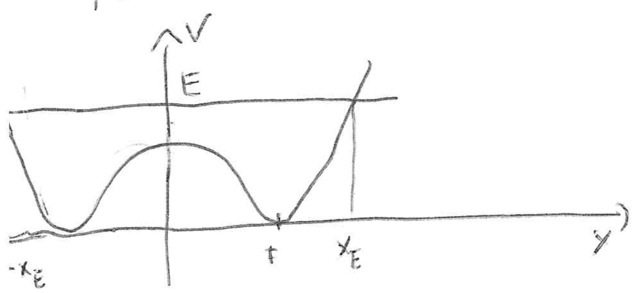
$$x(t) \simeq F + x_E^{MAX} \cos(\omega t)$$

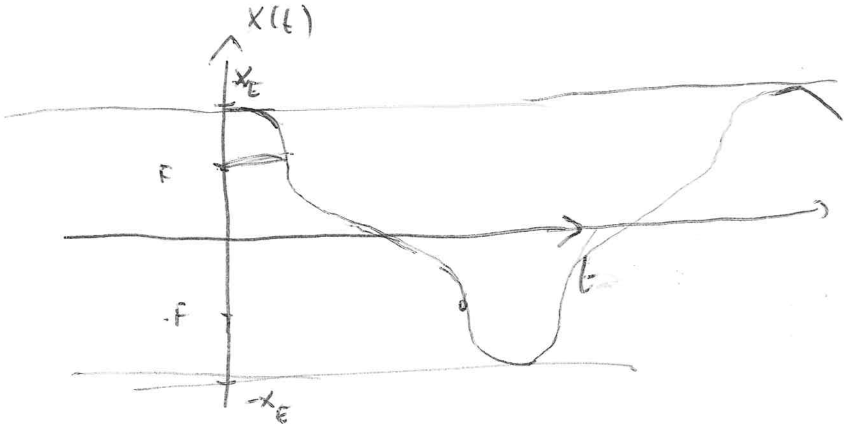
(ω is obtained by expanding $V(x)$ around



for $x \in (-x_E^{MIN}, -x_E^{MAX})$.

If $E > E_x$





Note, in both case there is "nothing like a soliton" here... there is just "t", you do not have a spatial solution...

$$L = \frac{1}{2} m_0 \dot{x}^2 - V(x) = \frac{1}{2}\left(\frac{d(\sqrt{m_0}\, x)}{dt}\right)^2 - V(x)$$

$x \mapsto X/\sqrt{m_0}$

$$L = \frac{1}{2} \dot{X}^2 - V(X/\sqrt{m_0}) \equiv \frac{1}{2} \dot{X}^2 - V(X)$$ → we can also set for sake of simplicity $m_0 = 1$.

Which are the dimensions now?

L must have dimension of E.

Then, $[x] = [E^{-1/2}]$.

$V(x) = \frac{1}{2} \omega^2 x^2$ → $[\omega]$ has indeed dimension energy, $[\omega] = [E]$

Note, ω is indeed the "mass" of the field, not m_0....

In fact, for the mass of a field we understand the frequency of its fluctuation over time.

(This is why we can set $m_0 = 1$...). (Note: $V(x) = \frac{1}{2} k x^2$

$k = m_0 \omega^2 \mapsto \omega = \sqrt{\frac{k}{m_0}}$, this is why m_0 is important for practical purposes...)

$$V(x) = \frac{\lambda}{4} (x^2 - F^2)^2$$

$[F] = E^{-1/2}$

$[\lambda] = E^3$

Note:

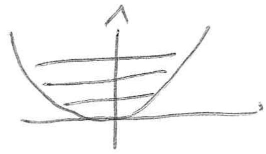
$V(x) = c x^n$ → $[c]$ has positive energy,... obviously, always "renormalizable"...

Nul

QM-recall

$$L = \frac{1}{2}\dot{x}^2 - V(x)$$

(a) $V(x) = \frac{1}{2}\omega^2 x^2$

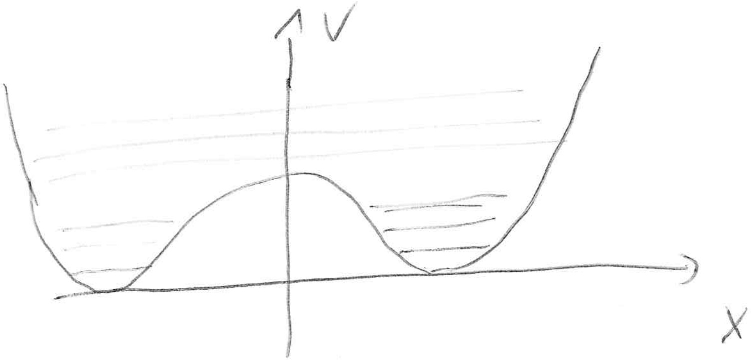


$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$E_n = \left(n+\tfrac{1}{2}\right)\omega$$

($V(x) = \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{4}x^4$ ... they are not equidistant, but you have a similar feature...).

(b) $V(x)$



How are the energy level here?

is there a degeneracy?

$E_n = (n+\frac{1}{2})\omega$ in the right well ??

$E_n = (n+\frac{1}{2})\omega$ " " left well ??

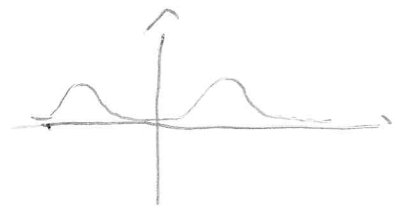
This is not the case... a general theorem actually forbids this kind of degeneracy for 1-dim systems.

[You would have it for:

separated wells, no communication, no tunnel... but you can't have it for a well-behaved potential]

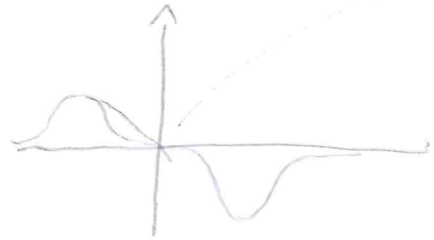
Indeed, a solution of the Schrödinger equation shows that

$\psi_S(x)$



$$E_S = \frac{\omega}{2} - \delta$$

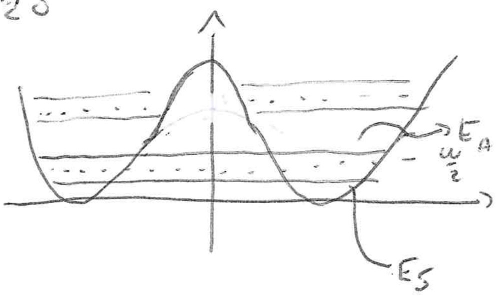
$\psi_A(x)$



$$E_A = \frac{\omega}{2} + \delta$$

(node $\to$ higher energy...)

$$E_A - E_S = 2\delta$$



$$V = \frac{\lambda}{4}(x^2 - F^2)^2$$

Close to E_* it can be then extremely complicated...

We have this solution because of tunnelling: I can go from left to right and viceversa.

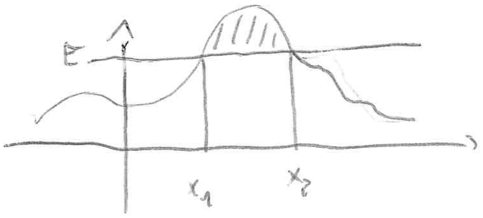
$$\begin{cases} \Delta E = E_A - E_S = 2\delta = \sqrt{\dfrac{6 S_0}{\pi}}\, e^{-S_0} \\[2ex] S_0 = \dfrac{\omega^3}{3\lambda} = \dfrac{m^3}{3\lambda} \end{cases}$$

$\to$ Non-perturbative (tunnel!)
(This is valid in the WKB-approx.)

$\boxed{\omega = m} \to$ "mass" of the fluctuation...

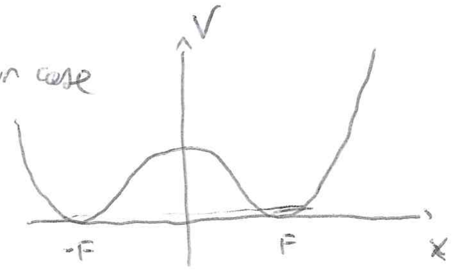
$$\omega^2 = \frac{\partial^2 V}{\partial x^2}\Big|_F = \frac{d}{dx}\Big(\lambda x (x^2 - F^2)\Big)_F = \Big(\lambda(x^2 - F^2) + 2\lambda x^2\Big)_F = 2\lambda F^2$$

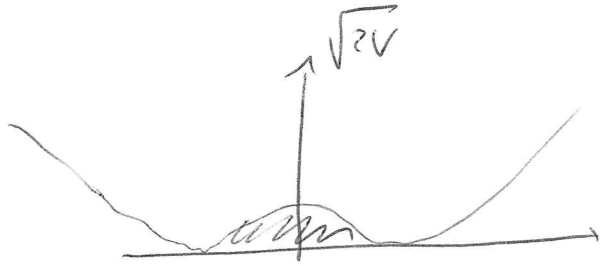
Tunnel: do you recall the tunnel probability?



$$T \approx e^{-\int_{x_1}^{x_2} dx \sqrt{2m_0 (V(x)-E)}}$$

In our case

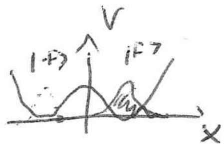




$$T \approx e^{-\int_{-F}^{F} dx \sqrt{2V(x)}} = e^{-S_0}$$

$\equiv$

Why is the symmetric configuration lower in energy than the other one?



$|\tilde{R}\rangle \equiv$ state localized on the right

$|\tilde{L}\rangle \equiv$ " " " " left

(Achtung: formally, $|F\rangle$ is:


)

Which is the approximate hamiltonian?

$$H = \frac{m}{2}|R\rangle\langle R| + \frac{m}{2}|L\rangle\langle L| - \delta\left(|R\rangle\langle L| + |L\rangle\langle R|\right)$$

$$\boxed{\delta \propto T \approx e^{-S_0} \ !!!}$$

$$|S\rangle = \sqrt{\tfrac{1}{2}}\,(|R\rangle + |L\rangle)$$

$$|A\rangle = \sqrt{\tfrac{1}{2}}\,(|R\rangle - |L\rangle)$$

Ergo:

$$\begin{cases} |R\rangle = \frac{1}{\sqrt{2}}(|S\rangle + |A\rangle) \\ |L\rangle = \frac{1}{\sqrt{2}}(|S\rangle - |A\rangle \end{cases} \qquad \begin{cases} \langle R| = \frac{1}{\sqrt{2}}(\langle S| + \langle A|) \\ \langle L| = \frac{1}{\sqrt{2}}(\langle S| - \langle A| \end{cases}$$

$$H = \frac{m}{2}\left(\frac{1}{2}|S\rangle\langle S| + \frac{1}{2}|A\rangle\langle A| + \frac{1}{2}|S\rangle\langle A| + \frac{1}{2}|A\rangle\langle S|\right)$$

$$+ \frac{m}{2}\left(\frac{1}{2}|S\rangle\langle S| + \frac{1}{2}|A\rangle\langle A| - \frac{1}{2}|S\rangle\langle A| - \frac{1}{2}|A\rangle\langle S|\right)$$

$$-\delta\Big[\frac{1}{2}(|S\rangle\langle S| - |S\rangle\langle A| + |A\rangle\langle S| - |A\rangle\langle A|)$$

$$+ \frac{1}{2}(|S\rangle\langle S| + |S\rangle\langle A| - |A\rangle\langle S| - |A\rangle\langle A|)$$

$$= \frac{m}{2}(|S\rangle\langle S| + |A\rangle\langle A| - \delta(|S\rangle\langle S| - |A\rangle\langle A|)$$

$$= \left(\frac{m}{2} - \delta\right)|S\rangle\langle S| + \left(\frac{m}{2} + \delta\right)|A\rangle\langle A|$$

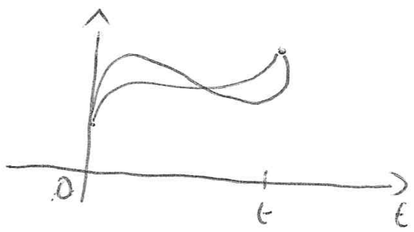
we see that the new basis $\{|S\rangle, |A\rangle\}$ diagonalises H ... and that for $\delta < 0$, $|S\rangle$ is lower than $|A\rangle$.

Instanton = link to solitons.

In QM, one may use the path integral approach

$L(t) = \frac{1}{2}\dot{x}^2 - V(x)$

$$\langle x_2 | e^{-iHt} | x_1 \rangle = \int_{\substack{x(0)=x_1 \\ x(t)=x_2}} Dx \, e^{+i\int_0^t dt_1 L(t_1)}$$



Now, let us consider the transformation

$\boxed{t = -iY}$ Euclidean time

The quantity

$$\langle x_2 | e^{-HY} | x_1 \rangle$$

is perfectly defined from a mathematical point of view:

$$\langle x_2 | e^{-HY} | x_1 \rangle = \int_{\substack{x(0)=x_1 \\ x(Y)=x_2}} Dx \, e^{-\int_0^Y dY_1 L^E(Y_1)}$$

where:

$$L^E(Y_1) = \frac{1}{2}\frac{dx^2}{dY^2} + V(x)$$

with reversed sign...

Note, this is only a mathematical object... still, it contains all the information about the quantum system.

In fact:

$$\langle x_2|e^{-HY}|x_1\rangle = \sum_{n,m} \langle x_2|n\rangle \langle n|e^{-HY}|m\rangle\langle m|x_1\rangle =$$

$$= \sum_n \langle x_2|n\rangle\langle n|x_1\rangle e^{-E_n Y}$$

For instance, if you make sure that $\langle x_2|n\rangle \neq 0$, $\langle n|x_1\rangle \neq 0$, and you perform the limit $Y = T \mapsto \infty$, you get:

$$\langle x_2|e^{-HT}|x_1\rangle \simeq \langle x_2|0\rangle\langle 0|x_1\rangle e^{-E_0 T} = \int_{\substack{x(-T/2)=x_1 \\ x(T/2)=x_2}} Dx\, e^{-S_E} \rightarrow e^{-E_0 T} = \frac{1}{\langle x_2|0\rangle\langle 0|x_1\rangle}\int_{BC} Dx\, e^{-S_E}$$

Recall also that the partition function

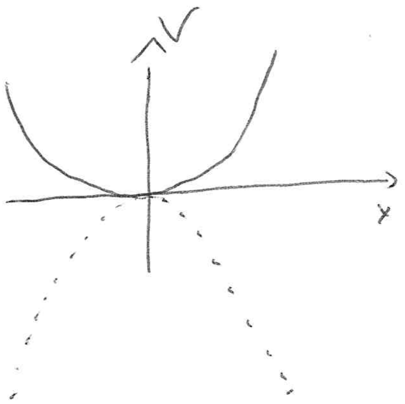
$$Z = \sum_n \langle n|e^{-H\beta}|n\rangle = \sum_n e^{-\beta E_n} = \int_{\substack{PBC \\ x(0)=x_1 \\ x(\beta)=x_1}} Dx\, e^{-\int_0^\beta dY L_E(Y)}$$

$$\boxed{E_0 = \lim_{T\mapsto\infty} \left\{ -\frac{1}{T} \cdot \frac{\int_{BC} Dx\, e^{-S_E}}{\langle x_2|0\rangle\langle 0|x_1\rangle} \right\}}$$

Still, where is "the solution"?

$$\int_{wBC} Dx \; e^{-\int_{-T/2}^{T/2} dY \left(\frac{1}{2}\frac{dx^2}{dY^2} + V(x)\right)}$$

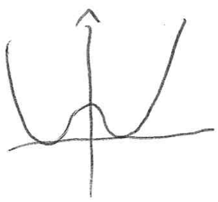
$$\frac{d^2x}{dY^2} = + \frac{dV}{dX}$$



We see that the classical relevant solution is $X_{cl}(Y) = 0$! No more...

The solution is in this case simple...

But now, suppose that you have the double-well:

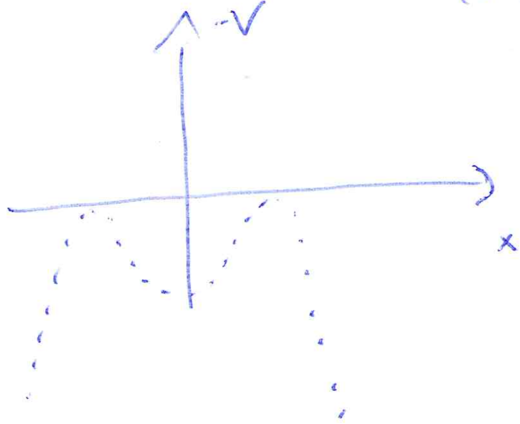


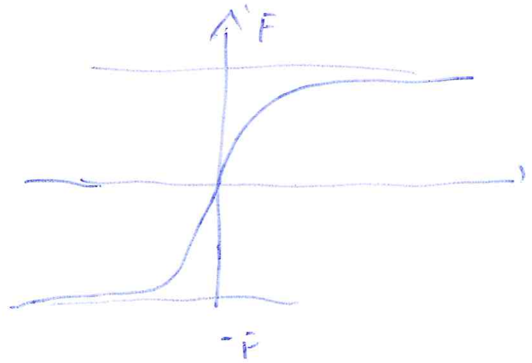
if $\langle F| e^{-HT} |F\rangle \mapsto X(Y) = F$ is a "good classical solution, with $S_{cl} = 0$.

But, if we aim to calculate

$$\langle F | e^{-HY} | -F \rangle = \int\limits_{\substack{x(-\frac{T}{2}) = -F \\ x(\frac{T}{2}) = F}} Dx \, e^{-S_E}$$

$(T \gg 1)$





$$x_{inst}(Y) = F \tanh\left(\frac{m}{2} Y\right)$$

$$S_E(x_{inst}) = \int L_E \, dY = \int \left[\frac{1}{2}\left(\frac{dx}{dY}\right)^2 + V(x)\right] dY = M_{KINK} =$$

[In fact: we minimize $S_E = \int \left[\frac{1}{2}\left(\frac{dx}{dY}\right)^2 + V(x)\right] dY$,

which means that the Lag. $L(t) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)$, which has

(Recall of the usual way:

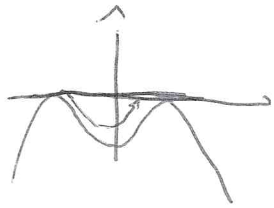
$L = \frac{1}{2}\dot{x}^2 - U$, $\quad \mathcal{E} = \frac{1}{2}\dot{x}^2 + U = \text{cnst}$, $\quad x = x(t)$ is a solution which extremizes the action

$S = \int L \, dt \ldots$

Now, we have the same with: $V = -U$, $t \mapsto Y$, $S \mapsto S_E$ (minimal for the motion with $\mathcal{E} = 0$)

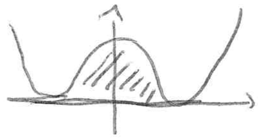
Note that:

$$S_E(X_{inst}) = \int_{-\infty}^{\infty} dY\, L_E = \int_{-\infty}^{\infty} dY \left[\frac{1}{2}\left(\frac{dx}{dY}\right)^2 + V(x)\right] = \int_{-\infty}^{\infty} dY \left(\frac{dx}{dY}\right)^2$$



$$\mathcal{E} = 0 = \frac{1}{2}\left(\frac{dx}{dY}\right)^2 - V(x)$$

$$\frac{1}{2}\left(\frac{dx}{dY}\right)^2 = V(x)$$



$$\int_{-\infty}^{\infty} dY \left(\frac{dx}{dY}\right)\left(\frac{dx}{dY}\right) = \int_{-F}^{F} \frac{dx}{dY}\, dx = \int_{-F}^{F} \sqrt{2V(x)}\, dx = M_{KINK} = S_{inst}$$

$$x = x_{inst}(Y) = F \tanh\left(\frac{m}{2} Y\right)$$

(instanton action = M_{KINK})

Recall: $|S\rangle, |A\rangle$. $|S\rangle$ = ground state. $|A\rangle$: first excited state.
$|S\rangle = |0\rangle$ $|A\rangle = |1\rangle$

$$\langle S|e^{-HT}|S\rangle = \sum_{m=0}^{\infty} |\langle S|m\rangle|^2 e^{-E_m T} = e^{-E_0 T}$$

$$\langle F|e^{-HT}|F\rangle = |\langle F|S\rangle|^2 e^{-E_0 T} + |\langle F|A\rangle|^2 e^{-E_1 T} + \dots$$

$$\langle -F|e^{-HT}|-F\rangle = |\langle -F|S\rangle|^2 e^{-E_0 T} + |\langle F|A\rangle|^2 e^{-E_1 T} + \dots$$

$$\left(\sqrt{\tfrac{1}{2}}\left(\langle F| + \langle -F|\right) e^{-HT}\right)\left(\sqrt{\tfrac{1}{2}}\left(|F\rangle + |-F\rangle\right)\right) = \frac{1}{2}\left|\left(\langle F| + \langle -F|\right)|S\rangle\right|^2 e^{-E_0 T}$$

$$\left(\sqrt{\tfrac{1}{2}}\left(\langle F| - \langle -F|\right) e^{-HT}\right)\left(\sqrt{\tfrac{1}{2}}\left(|F\rangle - |-F\rangle\right)\right) = \frac{1}{2}\left|\left(\langle F| - \langle -F|\right)|A\rangle\right|^2 e^{-E_1 T}$$

Namely:

$$\begin{cases} \left(\langle F| - \langle -F|\right)|S\rangle = 0 \\ \left(\langle F| + \langle -F|\right)|A\rangle = 0 \end{cases}$$

Ergo:

$$\begin{cases} E_0 = -\dfrac{1}{T} \ln \dfrac{\left(\langle F| + \langle -F|\right) e^{-HT} \left(|F\rangle + |-F\rangle\right)}{\left|\left(\langle F| + \langle -F|\right)|S\rangle\right|^2} \equiv E_S \\[2ex] E_1 = -\dfrac{1}{T} \ln \dfrac{\left(\langle F| - \langle -F|\right) e^{-HT} \left(|F\rangle - |-F\rangle\right)}{\left|\left(\langle F| - \langle -F|\right)|A\rangle\right|^2} \equiv E_A \end{cases}$$

$$E_1 - E_0 = E_S - E_A =$$

$$= -\frac{1}{T} \ln\left(\frac{1}{\vert\vert^2}\left(2\langle F\vert e^{-HT}\vert F\rangle - 2\langle F\vert e^{-HT}\vert -F\rangle\right)\right.$$

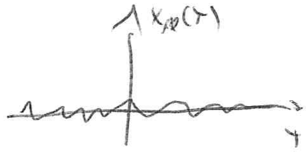
$$\left. + \frac{1}{T} \ln\left(\frac{1}{\vert\vert^2}\left(2\langle F\vert e^{-HT}\vert F\rangle + 2\langle F\vert e^{-HT}\vert -F\rangle\right)\right)\right.$$

$$\boxed{\Delta E \simeq \frac{1}{T}\, 2\, \frac{\langle F\vert e^{-HT}\vert -F\rangle}{\langle F\vert e^{-HT}\vert F\rangle}}$$

What to do to evaluate $\langle F | e^{-HT} | -F \rangle$ and then all the rest:

$S_E(x)$

$x = x_{inst}(\tau) + x_{fl}(\tau)$

$x_{fl}(\tau)$ 

$$S_E(x) \simeq S_E(x_{inst}) + \underbrace{\frac{\delta S_E}{\delta x}\Big|_{x_{inst}}}_{=0} (x_{fl}) + \frac{1}{2} \frac{\delta^2 S}{\delta x^2}\Big|_{x_{inst}} x_{fl}^2$$

One finds:

$$S\left(x_{inst} + \sum_n c_n x_n(\tau)\right) = e^{-S_0} \int_{-\infty}^{\infty} \prod_n \frac{dc_n}{2\pi} e^{-\lambda_n c_n^2}$$

$$\left[-\partial_\tau^2 + V''(x_{inst}(\tau))\right] x_n = \lambda_n x_n$$

'Problems' (∞) with translational invariance.

Sum over many instantons contributions...

Long way to get the correct result. Here we are already happy to have found e^{-S_0}!!!

Ergo, we learn that

Instanton: solution of the euclidean e.o.m. which connects different vacua.

Relevant tunnel probability... important for energy splitting.

Soliton: (in its 'ground state' or 'rest frame') spatial solution

$$\boxed{\text{Soliton in } 1+D \text{ dimensions} \longleftrightarrow \text{Instanton in } (1+(D-1)) \text{ dimensions}}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V \qquad \partial_\mu = (\partial_0, \partial_1, \dots, \partial_D)$$

$$\partial_t^2\phi - \Delta\phi = -\frac{\partial V}{\partial\phi}$$

Spatial solution

$$\Delta\phi(\vec{x}) = \frac{\partial V}{\partial\phi}$$

with finite energy (but nonzero)

$$E \equiv M_{soliton} = \int d\vec{x}\, \mathcal{E} = \int_{-\infty}^{\infty} d\vec{x} \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right]$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V \qquad \underbrace{\partial_\mu\, (\partial_0, \partial_1, \dots, \partial_{D-1})}$$

$$\mathcal{L}_E = -\frac{1}{2}(\partial_\mu^E\phi)^2 - V$$

solution of the e.o.m.

$$\Delta\phi(\vec{x}_E) = \frac{\partial V}{\partial\phi} \qquad \vec{x}_E = \begin{pmatrix} x_E^1 \\ \vdots \\ x^{D-1} \\ x^D = \tau \end{pmatrix}$$

with finite action (but nonzero)

$$S_E = S_{INST} = \int d\vec{x}_E\, \mathcal{L}_E = \int_{-\infty}^{\infty} d\vec{x}_E \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + V\right]$$

very same expression !!!!