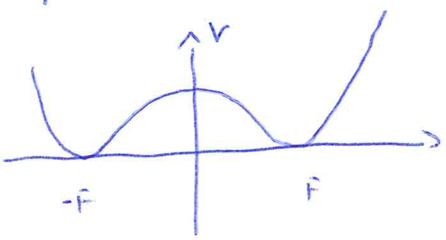
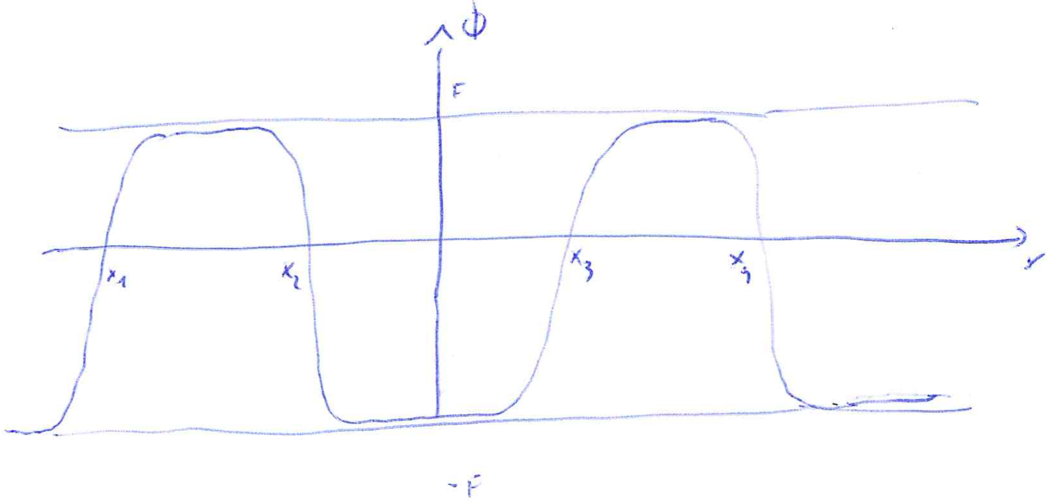
$$V = \frac{\lambda}{4}\left(\phi^2 - F^2\right)^2$$



$$m^2 = 2\lambda F^2 = \left.\frac{\partial^2 V}{\partial \phi^2}\right|_F$$

$\phi = F \tanh\left(\frac{m}{2}(x - x_1)\right)$ "static" soliton in x_1. $\phi = -F \tanh\left(\frac{m}{2}(x - x_2)\right)$: "static" antisoliton in x_2.

ϕ



only back and forward...

$$\phi(t,x) \overset{\sim}{=} F \tanh\left(\frac{m}{2}(x - x_1)\right) - F \tanh\left(\frac{m}{2}(x - x_2)\right)$$

$$+ F \tanh\left(\frac{m}{2}(x - x_3)\right) - F \tanh\left(\frac{m}{2}(x - x_4)\right) - F$$

$$\begin{cases} \phi(t, x \mapsto -\infty) = -F - F(-1) - F - F - F(-1) - F = -F \\ \phi(t, x \mapsto +\infty) = \end{cases}$$

Topological charge and current:

$$\begin{cases} J^\mu = \dfrac{1}{2F}\,\varepsilon^{\mu\nu}\partial_\nu \phi \\ Q = \displaystyle\int_{-\infty}^{\infty} dx\, J^0 = \dfrac{1}{2F}\left(\phi(x \to \infty) - \phi(x \to -\infty)\right) \end{cases}$$

$$\begin{cases} Q_{soliton} = 1 \\ Q_{Antisoliton} = -1 \end{cases}$$

$Q_{meson\text{-}fluct} = 0$ (top. trivial)

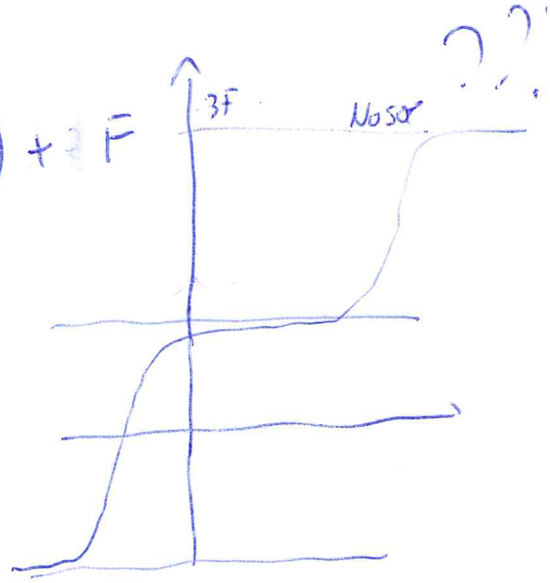
Note, this particular model allows only $Q = 0$, $|Q| = 1$.

In fact, if one makes the sum of two solitons the boundary conditions are not fulfilled:

$$\phi = F \tanh\left(\frac{m}{2}(x - x_1)\right) + F \tanh\left(\frac{m}{2}(x - x_2)\right) + F$$

$$\phi(t, x \to -\infty) = -F - F + F = -F \quad \checkmark$$

$$\phi(t, x \to +\infty) = F + F + F = 3F$$



this is not a minimum...

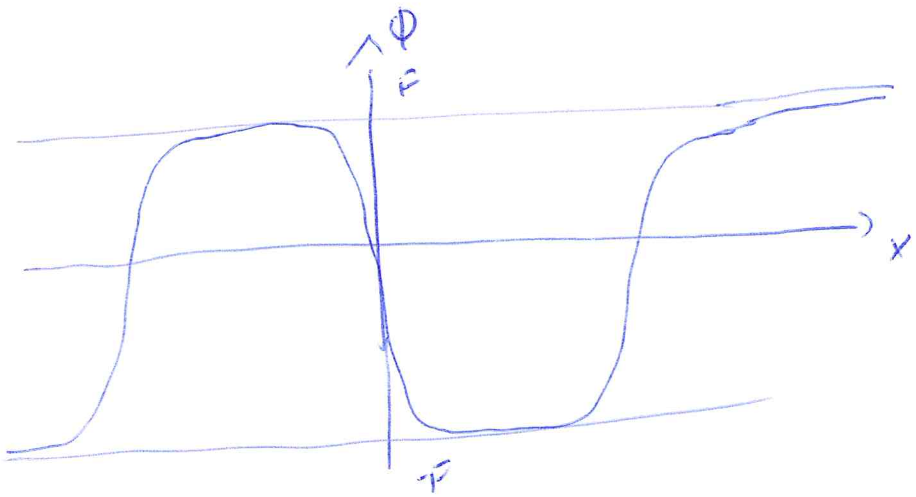
You can of course write down this object, but its energy is infinite!!!

So, this particular model is such that $|Q| \le 1$.

Bogomol'nyi inequality:

$\phi = \phi(x)$

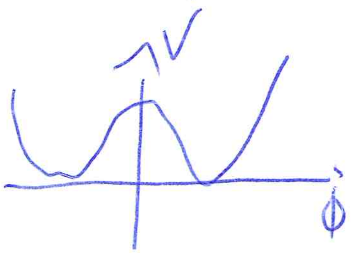
$$\begin{cases} E \geq M_{KINK} \cdot |Q| \\ M_{KINK} = \frac{2}{3} F^2 m . \end{cases}$$



$Q = 1$ (2 solitons, 1 antisolitons)

$E \simeq 3 M_{KINK}$ (actually the real calculation will be more complicated if you take into account the "interaction" between the solitons...

The very same qualitative properties hold for each model with a potential of the form



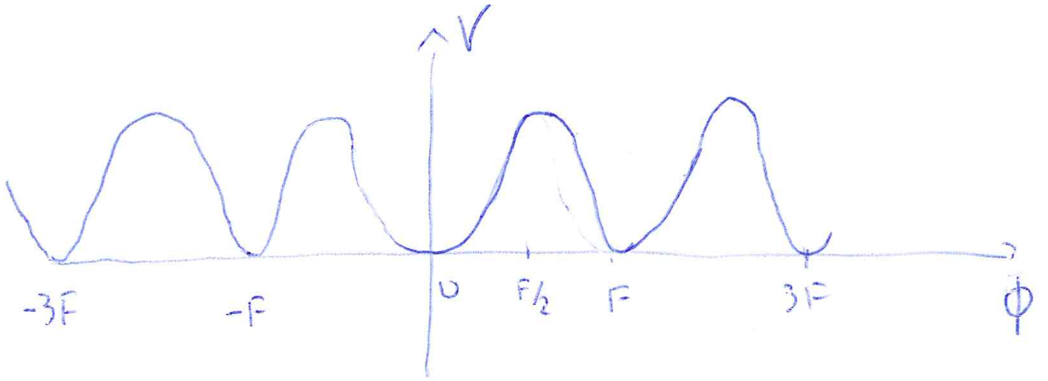
This is indep. on the details of the form.

There will be soliton, antisoliton, ... $|Q| \leq 1$, ...

Sine-Gordon-model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

$$V(\phi) = A\left[1 - \cos\left(\frac{2\pi\phi}{F}\right)\right] \qquad , \begin{cases} A > 0 \\ F > 0 \end{cases}$$



$$(\partial_t^2 - \partial_x^2)\phi = -\frac{\partial V}{\partial \phi} = -\frac{A 2\pi}{F}\sin\left(\frac{2\pi\phi}{F}\right)$$

$$(\partial_t^2 - \partial_x^2)\phi + \frac{2\pi A}{F}\sin\left(\frac{2\pi\phi}{F}\right) = 0$$ $\rightsquigarrow$ sine - Gordon (story of the name).

Taylor expansion of V around $\phi = 0$ implies:

$$V(\phi) = A\left[1 - 1 + \frac{1}{2}\left(\frac{2\pi\phi}{F}\right)^2 - \frac{1}{4!}\left(\frac{2\pi\phi}{F}\right)^4 + \dots\right]$$

$$= \frac{1}{2} A \left(\frac{2\pi}{F}\right)^2 \phi^2 - \underbrace{\frac{1}{4!}\left(\frac{2\pi}{F}\right)^4}_{\text{perturbative attraction}} \phi^4 + \dots$$

Small fluctuation around $\phi = 0$ $\longmapsto$ particles with mass

$$m^2 = A\left(\frac{2\pi}{F}\right)^2 ;$$

Notation: 't Hooft vs Rosenoran

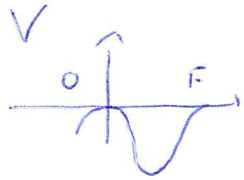
$$\begin{cases} A = \dfrac{m^4}{\lambda} \\ \dfrac{2\pi}{F} = \dfrac{\sqrt{\lambda}}{m} \end{cases}$$

$(A, F) \longleftrightarrow (m, \lambda)$.

check:

$$A \cdot \frac{4\pi^2}{F^2} = \frac{m^4}{\lambda} \frac{4\pi^2}{\left(\frac{2\pi m}{\sqrt{\lambda}}\right)^2} = m^2 \quad \checkmark \text{ q.ed}$$

This theory admits solitons:

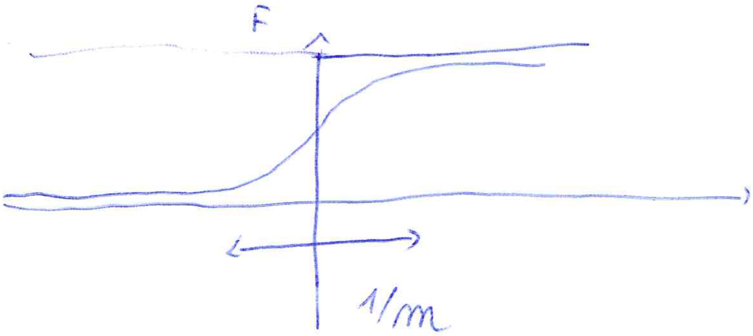


$\phi(x)$ "spatial only

$$\phi(x) = \frac{2F}{\pi}\arctan\left(e^{mx}\right)$$

$$\phi(x \mapsto -\infty) = \frac{2F}{\pi}\arctan(0) = 0$$

$$\phi(x \mapsto +\infty) = \frac{2F}{\pi}\cdot\frac{\pi}{2} = F$$



$$\phi(x) = \frac{2F}{\pi}\arctan\left(e^{m(x-x_0)}\right)$$

By "boosting" we can get moving solitons. Very similar analysis as the ϕ^4-case in that respect.

$$\phi(x) = -\frac{2F}{\pi}\arctan\left(e^{m(x-x_0)}\right)$$ is an antisoliton.

$$J^\mu = \frac{1}{2F}\,\varepsilon^{\mu\nu}\partial_\nu\phi$$ is the top. current.

Q is the top. charge. $Q = 0$ for "mesons"

$Q = 1$ for one soliton

$Q = -1$ for one antisoliton.

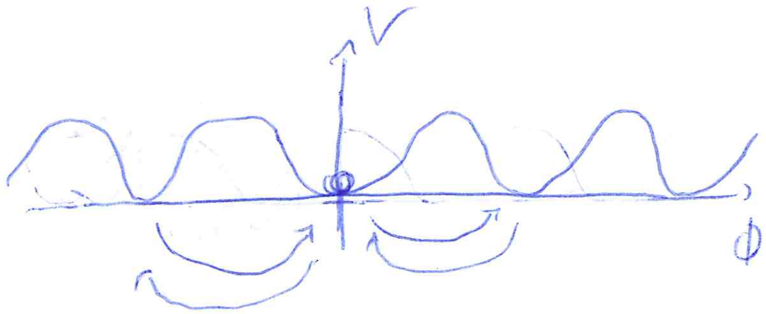
But now there are also new elements:

$$\phi(x) = F\arctan\left(e^{m(x-x_0)}\right) - F$$

$\phi(x \to -\infty) = -F$

$\phi(x \to +\infty) = 0$

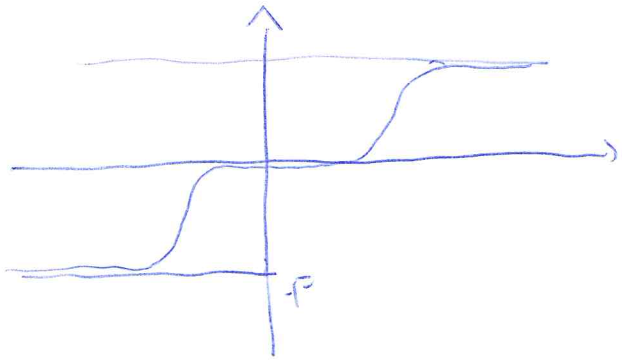
is also a solution with finite energy.



$$\phi(x) = F\arctan\left(e^{m(x-x_0)}\right) - nF$$

$n = 0, \pm 1, \pm 2, \ldots$ is then a solution.

Now we can also construct more complicated solutions...

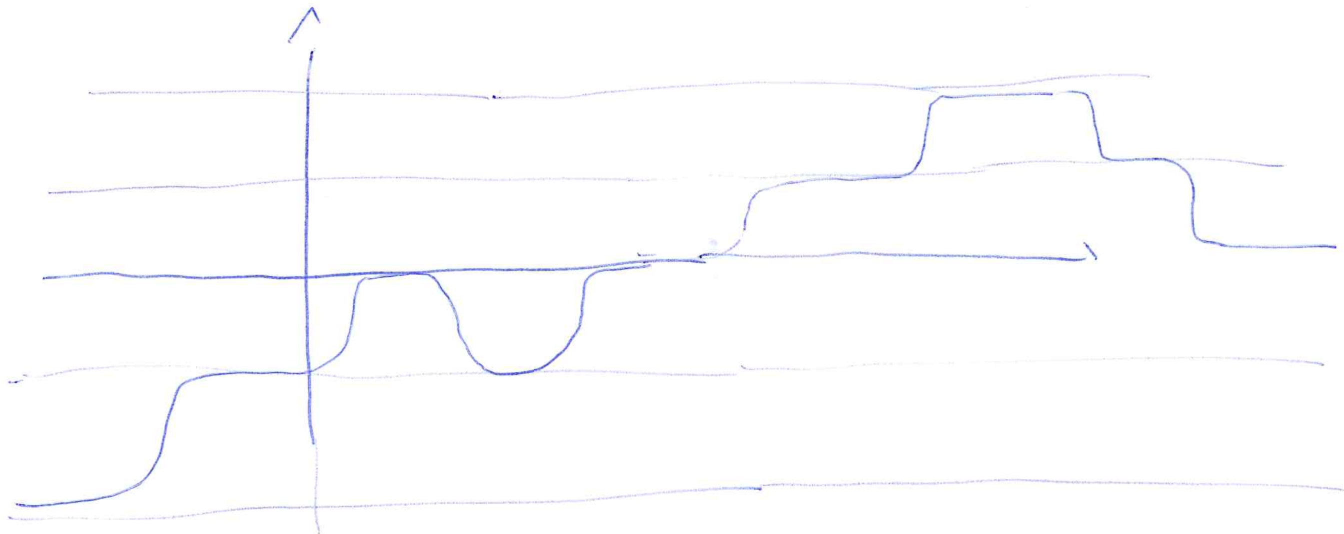


$Q = 2$

$$\boxed{E \geq M_{soliton} |Q|}$$

$Q = 0, \pm 1, \pm 2, \ldots$

Namely, it can be much more complicated...



This has also $Q = 2$.

Indeed, exact solutions can be written down in this case.

S-A scattering (attraction when they are close)

S-S
A-A | (repulsion when they are close)

Breather: S-A bound state

⇝ possible subject of one short presentation (see Rajaraman). The best would be: plot $E(t,x)$ as function of t (or do a movie ...).

⇝ $U(\phi_1, \phi_2)$ is another possibility for a short presentation. Also, a short numerical analysis would be useful.

Solitons: when do they exist?

Scalar field theory in $1+D$ dimensions. Which condition must D fullfill in order to have solitonic solutions?

At first, one may answer: such solution exist for each D... why should some D be excluded?

'Naive' thinking:

$$V = \frac{\lambda}{4}\left(\phi^2 - F^2\right)^2 \quad \text{in } 1+2 \text{ dimensions.} \quad \phi(t,x,y).$$

$$\phi(t,x,y) = F \tanh\left(\frac{m}{2}(x-x_0)\right)$$

This is still a solution of the e.o.m. (The derivative $\partial_y \phi = 0$ ...).

But: $E = \infty$.

So, the crucial point is to have a solution of the e.o.m. with finite energy.

And, as we shall show, this is possible only if D is small enough.

$D=1$? $\to$ such sol. solution exist, we have proven it.

But what about $D=2$, or $\underbrace{D=3}_{\text{our world}}$? Which is the maximal D?

D spatial dimensions. $\overset{\text{time}}{1} + D$ dimensions.

$\partial_\mu \qquad \mu = 0, 1, \ldots, D$

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$ N scalar fields.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi})^2 - U(\vec{\phi}) \longrightarrow (\partial_t^2 - \Delta)\vec{\phi}_i = +\frac{\partial \mathcal{L}}{\partial \phi_i} = -\frac{\partial U}{\partial \phi_i}$$

Note:

$\vec{\phi} = \vec{\phi}(t, \vec{x})$ where $\vec{x} = \begin{pmatrix} \vec{x}_1 \\ \vdots \\ \vec{x}_D \end{pmatrix}$.

Let us now consider a spatial solution:

$\phi = \phi(\vec{x})$. (No t is present...).

$$\boxed{\Delta \phi_i = \frac{\partial U}{\partial \phi_i}}$$

We assume, without loss of generality that $U \geq 0$.

(In fact, U must be bounded from below, we can then easily shift it by a constant in such a way that the minimum coincides with zero).

The energy of this field configuration is

$$E = \int d^D x \left[\frac{1}{2} \left(\vec{\nabla} \vec{\phi} \right)^2 + U(\vec{\phi}) \right] = V_1[\vec{\phi}] + V_2[\vec{\phi}] .$$

$$V_1 \geq 0, \quad V_2 \geq 0$$

Now, let us suppose that there is a solitonic static solution $\vec{\phi}_{sol}(\vec{x})$; That means: $\vec{\phi}_{sol}(\vec{x})$ is a solution of e.o.m. and has finite energy M_{sol}

Then, let us consider the following fields:

$$\phi_\lambda(\vec{x}) = \vec{\phi}_{sol}(\lambda \vec{x}) . \qquad \lambda \in R$$

The energy of $\phi_\lambda(\vec{x})$ for a certain λ is

$$V_1 \left(\vec{\phi}_{sol}(\lambda \vec{x}) \right) = \int d^D x \left[\frac{1}{2} \left(\vec{\nabla} \vec{\phi}_{sol}(\lambda \vec{x}) \right)^2 \right]$$

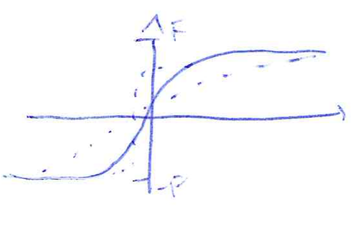
$$= \lambda^{2-D} V_1 \left(\vec{\phi}_{sol}(\vec{x}) \right)$$

$\vec{x}\,' = \lambda \vec{x}$

$\vec{x} = \frac{1}{\lambda} \vec{x}\,'$

$d^D x = \lambda^{-D} d^D x'$

$\vec{\nabla}_{\vec{x}} = \lambda \vec{\nabla}_{\vec{x}\,'}$



$\Rightarrow$ same topology, different form (width!)

We do the same with the second piece:

$$V_2\left(\vec{\phi}_\lambda(\vec{x})\right) = \lambda^{-D} V_2\left(\vec{\phi}_{sol}(\vec{x})\right)$$

Ergo:

$$E(\lambda) = \lambda^{2-D} V_1\left[\phi_{sol}(\vec{x})\right] + \lambda^{-D} V_2\left(\phi_{sol}(\vec{x})\right)$$

Now, it must be that $E(\lambda)$ has a minimum for $\lambda = 1$.

Namely:

If there are solution with smaller energy, we would have another solution...

The transf. $\vec{x} \mapsto \lambda \vec{x}$ does not change the topology.

Note also that:

$$\begin{cases} E = \int d^D x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + U\right] \\ \delta E = 0 \Longleftrightarrow \Delta\phi_i = \dfrac{\partial U}{\partial \phi_i} \end{cases}$$

That is, E must have a minimum for the solution $\vec{\phi}_{sol}$.

$$\frac{dE}{d\lambda} = (2-D)\lambda^{2-D+1} V_1 - D\lambda^{-D+1} V_2 = 0 \text{ for } \lambda = 1.$$

$$(2-D) V_1(\vec{\phi}_{sol}) = D V_2(\vec{\phi}_{sol}).$$

But for $D > 2$ this is not possible, because the first term is negative.

Ergo, we learn that:

- $D > 2$ $(D = 3, 4, \dots)$

No $\vec{\phi}_{sol}(\vec{x})$ exist!!!

(σ-model in 1+3 ⟼ no solitons are hidden there, as long as only scalar and pseudoscalar fields are considered (and if no Skyrmion is considered!))

- Achtung: one may object:

$V_1 = 0$

$V_2 = 0$

would still be ok...

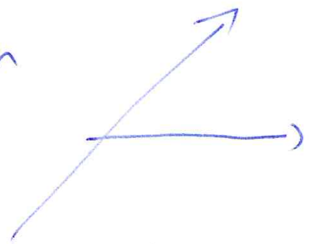
But $V_1 = 0$, $\vec{\phi}_{sol}(\vec{x}) = \vec{\phi}_{sol} = \text{const}$, $V_2(\vec{\phi}_{sol}) = 0 \dots$

This is a trivial solution with zero energy.

⟹ Here, solitons exist in 1+3... but other fields (such as vector fields) are needed. (QCD: gluon fields)

⤳ wonderful example of mathematics: by trying to write down some model for $D > 2$ with solitons, we would have not managed...

D = 2 Holomorphic



$V_2(\vec{\phi}_{sol}(\vec{x})) = 0$ everywhere ...

So, if there are many minima, they must be connected.

We will see an example of this type.

D = 1

$$\boxed{V_1(\vec{\phi}_{sol}(\vec{x})) = V_2(\vec{\phi}_{sol}(\vec{x}))}$$

This was indeed the case for an example.