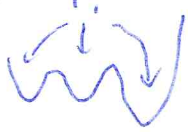
THE KINK / part II

Recall:

- $\phi(t,x) : \mathbb{R}^2 \mapsto \mathbb{R}$ is a scalar field in 1+1 dim
- $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - V(\phi)$ is the Lag $\Rightarrow$ $(\partial_t^2 - \partial_x^2)\phi = -\partial_\phi V$ is the e.o.m
- $\mathcal{E} = \mathcal{H} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + V(\phi)$ is the energy density...
- $E = \int_{-\infty}^{\infty} dx\, \mathcal{E}$ is the total energy of a certain field configuration
- $\frac{dE}{dt} = 0$ if $\phi(t,x)$ is a solution of e.o.m.

Note, a necessary condition for E finite is that $\phi(t, x \mapsto \pm\infty) = \phi_i$



- $V = \frac{1}{2} m^2 \phi^2$
  - $\to \phi(t) \to$ plane wave, $E = \infty$
  - $\to \phi(t,x) \to$ boost and w.p. $\to$ spread
  - $\to \phi(x) \to$ No solution

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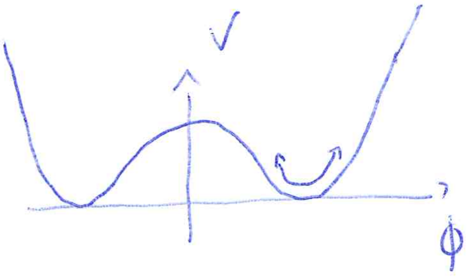
$$V = \frac{\lambda}{4}(\phi^2 - F^2)^2$$

$\boxed{\lambda > 0 \,,\, F > 0}$



$$m^2 = \frac{\partial^2 V}{\partial \phi^2}\bigg|_{\phi = F}$$

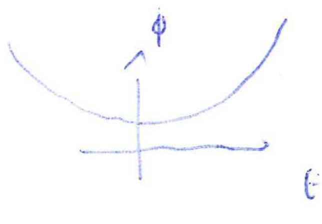
$\frac{\partial V}{\partial \phi} = \lambda \phi (\phi^2 - F^2)$, $\frac{\partial V}{\partial \phi}\Big|_F = 0$ (it is a minimum ...)

$\frac{\partial^2 V}{\partial \phi^2} = \lambda(\phi^2 - F^2) + \lambda \phi \cdot 2\phi \;\to\; m^2 = \frac{\partial^2 V}{\partial \phi^2}\Big|_F = 2\lambda F^2 \;\Rightarrow\; \boxed{m^2 = 2\lambda F^2}$

Note (1): if you would treat this theory perturbatively, then you get particles with mass m ($\phi \mapsto F + \phi$, then quantize...). The basic "energy" is then the mass m.

Note (2): if you try to quantize this theory around $\phi = 0$ you get a problem. $\frac{\partial^2 V}{\partial \phi^2}\Big|_{\phi = 0} = -\lambda F^2 < 0$ $\qquad \mu^2 = -\lambda F^2 < 0$

One gets a "negative" mass-squared. This is a tachyon... it means that you have picked up the "wrong" point to start the expansion.

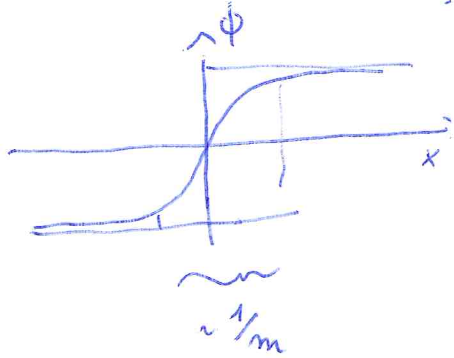


$\phi \sim \cos(\mu t) = \cos(i\sqrt{\lambda} F t)$

is not a plane wave...

(Digression on the neutrino and OPERA).

Full e.o.m:



$$\boxed{(\partial_t^2 - \partial_x^2)\phi = -\lambda\phi(\phi^2 - F^2)}$$

$$\boxed{\phi = \phi(x) = F\tanh\left(\frac{m}{2}x\right)} = F\,\frac{e^{\frac{m}{2}x} - e^{-\frac{m}{2}x}}{e^{\frac{m}{2}x} + e^{-\frac{m}{2}x}}$$

is a solution (good guess).

$$\partial_x\phi = F\frac{m}{2}\,\frac{1}{ch^2\left(\frac{m}{2}x\right)}$$

$$\partial_x^2\phi = F\frac{m^2}{4}\cdot\left(-\frac{2\,sh\left(\frac{m}{2}x\right)ch\left(\frac{m}{2}x\right)}{ch^3\left(\frac{m}{2}x\right)}\right) = -\frac{Fm^2}{2}\,\frac{\tanh\left(\frac{m}{2}x\right)}{ch^2\left(\frac{m}{2}x\right)}$$

Plug in the e.o.m:

$$+\frac{Fm^2}{2}\,\frac{\tanh\left(\frac{m}{2}x\right)}{ch^2\left(\frac{m}{2}x\right)} = -\lambda F\tanh\left(\frac{m}{2}x\right)F^2\left(\tanh^2\left(\frac{m}{2}x\right) - 1\right) =$$

$$= -\lambda F^3\tanh\left(\frac{m}{2}x\right)\frac{sh^2\left(\frac{m}{2}x\right) - ch^2\left(\frac{m}{2}x\right)}{ch^2\left(\frac{m}{2}x\right)} = \lambda F^3\,\frac{\tanh\left(\frac{m}{2}x\right)}{ch^2\left(\frac{m}{2}x\right)}$$

$$\frac{Fm^2}{2} = \lambda F^3 \qquad \boxed{m^2 = 2\lambda F^2}$$

(but this is exactly the definition of m...

q.e.d.

Then, we have found a solution which is spatial-dependent (only): if it has finite energy it is a soliton.

The energy density is

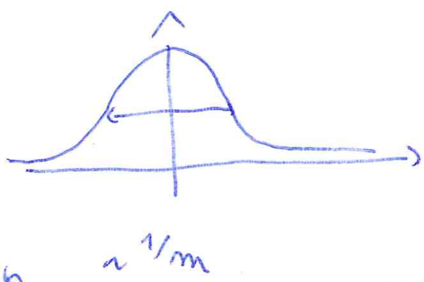
$$\mathcal{E} = \frac{1}{2}(\partial_x \phi)^2 + V(\phi) =$$

$$= \frac{1}{2}\left(F\frac{m}{2}\frac{1}{ch^2(\frac{m}{2}x)}\right)^2 + \frac{\lambda}{4}(\phi^2 - F^2)^2 =$$

$$= \frac{1}{2}\frac{F^2 m^2}{4}\frac{1}{ch^4(\frac{m}{2}x)} + \frac{\lambda}{4}F^4\frac{1}{ch^4(\frac{m}{2}x)}$$

$$m^2 = 2\lambda F^2$$

$$\mathcal{E} = \frac{\lambda F^4}{2}\frac{1}{ch^4(\frac{m}{2}x)} = \frac{F^2 m^2}{4}\frac{1}{ch^4(\frac{m}{2}x)}$$



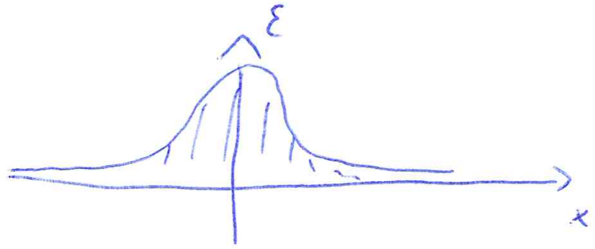
$$E \equiv M_{kink} = \int_{-\infty}^{\infty}\mathcal{E}\,dx = \frac{F^2 m^2}{4}\int_{-\infty}^{\infty}\frac{1}{ch^4(\frac{m}{2}x)}dx = \frac{F^2 m^2}{4}\cdot\frac{2}{m}\int_{-\infty}^{\infty}\frac{1}{ch^4(y)}dy = \frac{F^2 m}{2}\underbrace{\int_{-\infty}^{\infty}\frac{dy}{ch^4 y}}_{\text{this is a number}}$$

$$y = \frac{m}{2}x$$

$$\int_{-\infty}^{\infty}\frac{1}{ch^4 y}dy = \frac{1}{3}$$

Ergo:

$$\boxed{E \equiv M_{KINK} = \frac{2}{3} F^2 m}$$



Being $m = \sqrt{2\lambda}\, F$ we have

$$M_{KINK} = \frac{2}{3} F^3 \sqrt{2\lambda} \left(= \frac{2m^3}{\lambda} \right)$$

peculiar characteristic...

Suppose that:

- λ is small, F is large in such a way that $m = \sqrt{2\lambda}\, F^2$ is a finite number.

Actually being $[\lambda] = [\text{Energy}^2]$ we assume that $m \gg \sqrt{\lambda} \Rightarrow \underline{F \gg 1}$

Then, one sees that:

$$M_{kink} \gg m \;!$$

This non-perturbative solution carries much more energy than a single, perturbative particle. (Can the soliton decay to particles of mass m? The answer is: No! Later more on this).

Digression:

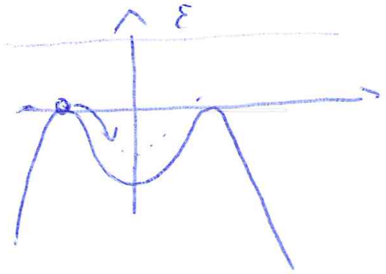
Guessing? In general yes... but actually in this case not quite.

$$(\partial_t^2 - \partial_x^2)\phi = -\partial_\phi V$$

$\phi = \phi(x)$ $\boxed{\partial_x^2 \phi = \partial_\phi V}$ → You would get this eq. of motion from

$$L = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)$$

(instead of t, you have x...
" " V, you have $-V$)



$$\phi(x) = F \tanh\left(\frac{m}{2}x\right)$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - V(\phi)$$

Cons. $\mathcal{E} = 0$

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = V(\phi) \qquad \frac{d\phi}{dx} = \pm\sqrt{2V(\phi)} \overset{(+)}{\longmapsto} \frac{d\phi}{\sqrt{2V(\phi)}} = dx \longmapsto \phi = F\tanh\left(\frac{m}{2}x\right)$$

u.e. solution!

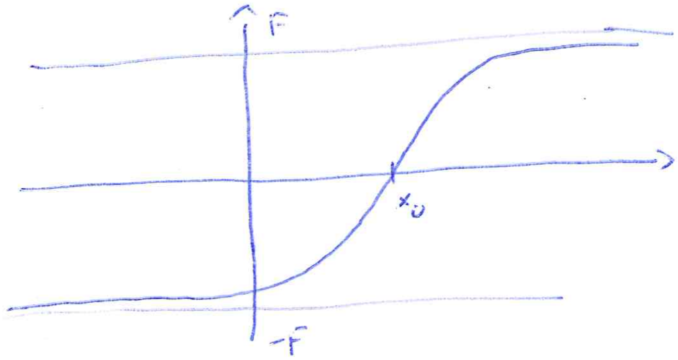
verify:

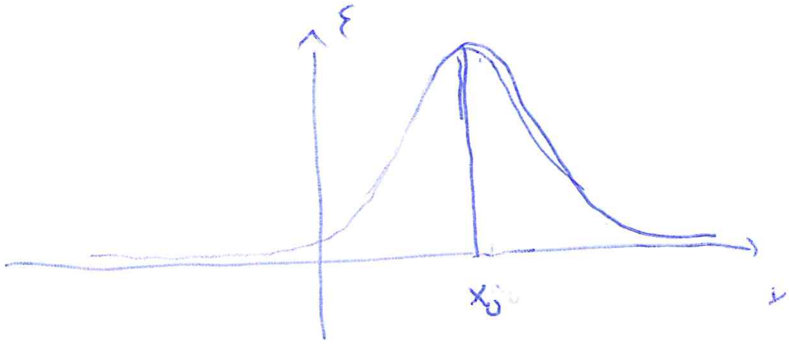
$$\mathcal{E} = \frac{1}{2}\left(\frac{m}{2}F\frac{1}{\cosh^2(\frac{m}{2}x)}\right)^2 - \frac{\lambda}{4}F^2\left(\tanh^2\left(\frac{m}{2}x\right) - 1\right)^2 = 0! \text{ q.e.d.}$$

How to get other soliton solutions

→ translation

$$\phi(x) = F \tanh\left(\frac{m}{2}(x - x_0)\right)$$





$$\mathcal{E} = \frac{F^2 m^2}{4} \frac{1}{\mathrm{ch}^4\left(\frac{m}{2}x\right)}$$

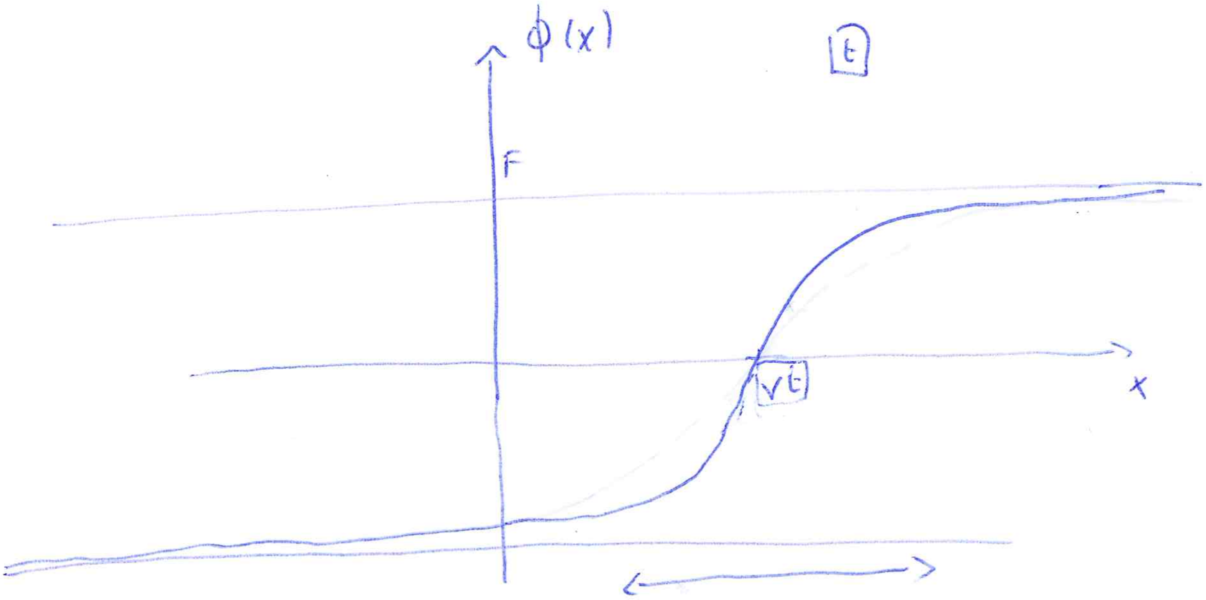
$$M = M_{kink} = \frac{2}{3} F^2 m .$$

→ boost

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} ; \quad \gamma = \frac{1}{\sqrt{1-v^2}} , \quad v \in (-1,1) .$$

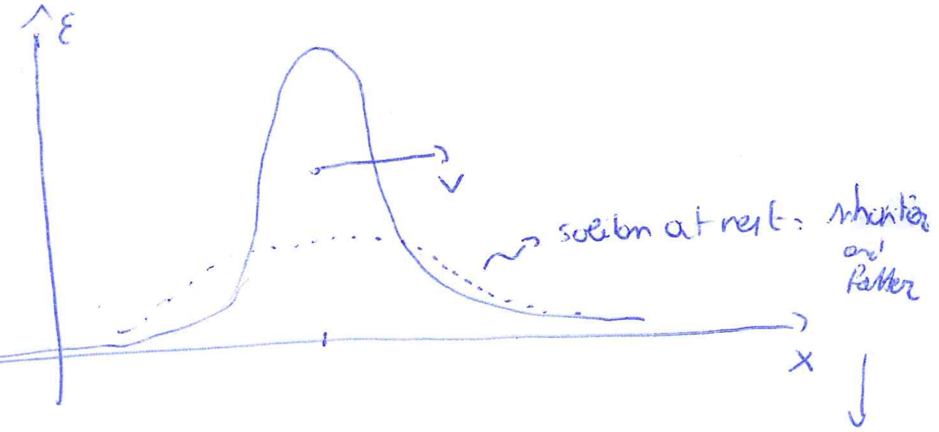
$$\phi(x') = F \tanh\left(\frac{m}{2} x'\right) \equiv F \tanh\left(\frac{m}{2}(-\gamma v t + \gamma x)\right) =$$

$$\boxed{\phi(t,x) = F \tanh\left(\frac{m}{2}\gamma(x - vt)\right)}$$



$$\frac{1}{\gamma m} < \frac{1}{m} \quad \text{because } \gamma > 1 .$$

$$\mathcal{E}(t,x) = \frac{F^2 m^2 \gamma^2}{4} \frac{1}{\mathrm{ch}^4\left(\frac{m}{2}\gamma(x-vt)\right)}$$



(n.b.: these are relat. results)

$$E_{\mathrm{kink}}(v) = \frac{2}{3} F^2 m \gamma = \gamma M_{\mathrm{KINK}}$$

In the end, the most general kink is given by

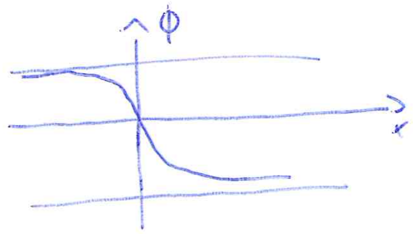
$$\phi(t,x) = F\tanh\left(\frac{m}{2}\gamma\left(x-x_0-vt\right)\right)$$

$$\begin{cases} x_0 \in \mathbb{R} \\ v \in (-1,1). \end{cases}$$

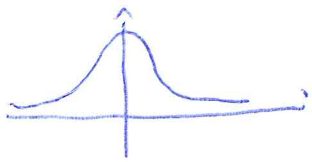
For each choice of x_0 and v we have a different kink...

Is it over? Not really. There is still a possible operation.

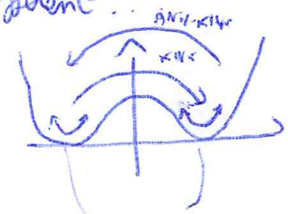
$$\boxed{\phi(x) = -F\tanh\left(\frac{m}{2}x\right)}$$



$\mathcal{E}(x)$ is just as the kink



Same $\mathcal{E}$, still they are different...



perturbative fluctuations

M " " " " "

Generalisation:

$$\phi(t,x) = -F\tanh\left(\frac{m}{2}\gamma(x-x_0-vt)\right)$$

This is called "anti-kink" or anti-soliton.

Digression:

- Mathe-file
- Film
- History
- Themes for a short presentation

Topological current and charge

$$J^\mu = \frac{1}{2v}\,\varepsilon^{\mu\nu}\,\partial_\nu \phi \qquad \varepsilon^{\mu\nu} = \begin{cases} \varepsilon^{01} = -\varepsilon^{10} = 1 \\ \varepsilon^{00} = \varepsilon^{11} = 0 \end{cases} \quad \mu,\nu = 0,1$$

where v is a constant.

Is this current conserved?

$$\partial_\mu J^\mu = \partial_0 J^0 + \partial_1 J^1 = \frac{1}{2v}\Big(\partial_0 \underbrace{\varepsilon^{01}}_{1}\partial_1\phi + \partial_1 \underbrace{\varepsilon^{10}}_{-1}\partial_0\phi\Big) =$$

$$= \frac{1}{2v}\Big(\partial_0\partial_1\phi - \partial_1\partial_0\phi\Big) = \frac{1}{2v}\Big(\partial_t\partial_x\phi - \partial_t\partial_x\phi\Big)$$

$$= 0 \;!!$$

Ergo, this current is conserved. Comments:

- But, it is always confirmed, for each ϕ ...
- No use of the e.o.m. (i.e. $\mathcal{L}$) or relation to a system.
- This conservation is true for each theory in the (1+1) Universe.

It is therefore called "topological" because it has to do with the form and the properties of the space we are working in, and not on the interaction.

You may say: "that's trivial... what do I learn from such a conservation?
If that current is conserved always, also for ϕ which are not solution of the e.o.m., what is the use of that?"

Answer: the charge.

Let us look at the conserved charge of the current:

$$Q = \int_{-\infty}^{\infty} J^0 dx = \frac{1}{2V}\int_{-\infty}^{\infty} (\partial_1 \phi)\, dx = \frac{1}{2V}\int_{-\infty}^{\infty} \partial_x \phi \, dx =$$

$$= \frac{1}{2V}\Big(\phi(x \mapsto +\infty) - \phi(x \to -\infty) \Big)$$

- For  $\to \phi(x \mapsto \pm\infty) = 0$. (The precise form of V is irrelevant!)
- For a generic field which describes small oscillations around one minimum we have also for :

$$\phi(x \mapsto +\infty) = \phi(x \mapsto -\infty) = F \to Q = 0$$

Let us now consider the soliton:

$$\phi = F \tanh\left(\frac{m}{2}(x-x_0)\right)$$

$$Q = \frac{F}{2v}\left(1-(-1)\right) = \frac{F}{2v}2 = \frac{F}{v}.$$

Note that, if in this case we 'normalise' by choosing $v=F$, we get:

$Q_{soliton} = +1$!

($Q_{soliton} = \pm 1$ even if the soliton is 'moving'.)

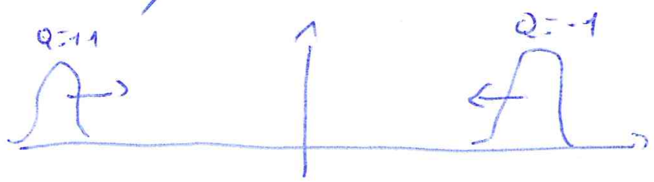
The soliton cannot decay in "mesonic fluctuation" because Q is conserved.

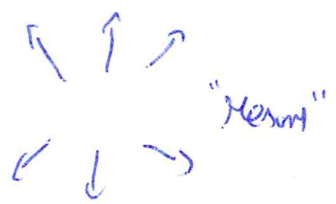
$Q=1$ shows that this is field configuration is indeed stable.

Antisoliton: $Q = -F \tanh\left(\frac{m}{2}(x-x_0)\right)$

$Q_{Antisoliton} = -1$

Eventually, a soliton + antisoliton solution could be an interesting case.





But nothing:

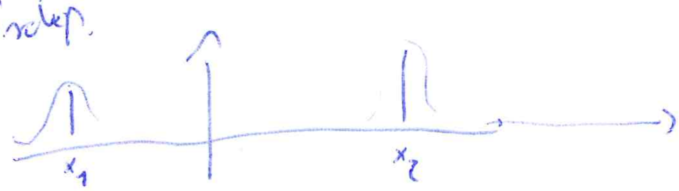
$\phi = \phi_{sol} + \phi_{antisol}$

is not a solutions because the e.o.m. is non-linear.

Still, if for $t \mapsto -\infty$ the wave-packets are very far, I can try to build a solution (approximate) of the type

$$\phi_{SA} = \phi_S + \phi_A - F$$

Example: t-indep.

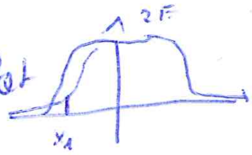


$$\phi_S = F \tanh\left(\frac{m}{2}(x - x_1)\right)$$

$$\phi_A = -F \tanh\left(\frac{m}{2}(x - x_2)\right)$$

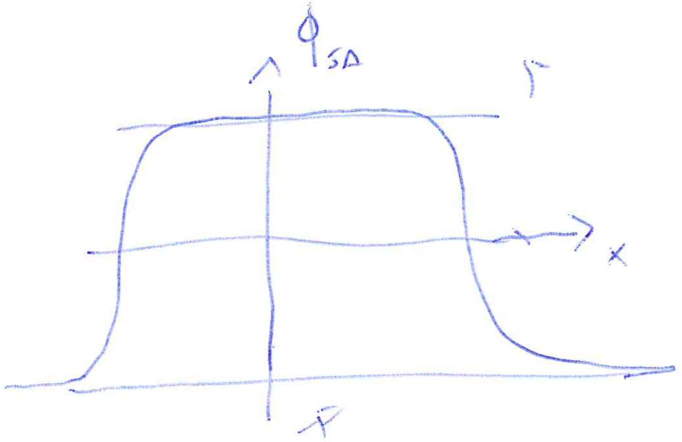
[nothing

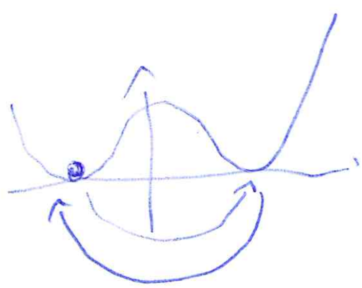
$\phi_S + \phi_A$ is such that



$\to 0$ for $x \mapsto \pm\infty$..

but this doesn't finite energy]

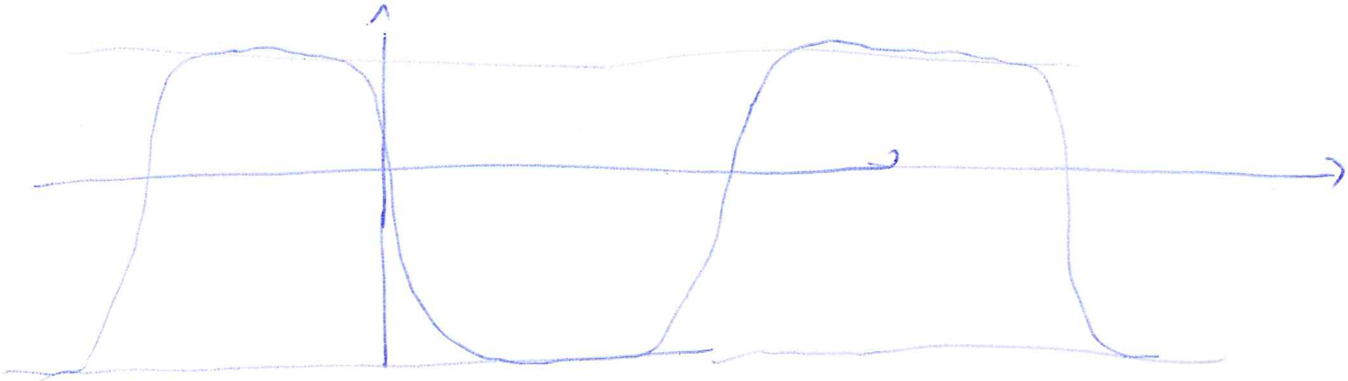




$$\boxed{Q = 0}$$

(of course, I could go on and consider v .. then it gets more complicated.)
(scattering)

or, I can also build a more difficult solution with



As long as they all well separated, all is "fine" ... you have an approximate solution with

$Q_{tot} = 0$.

Bogomol'nyi inequality

$\phi(x)$ is spatial only

$$\mathcal{E}(x) = \frac{1}{2} (\partial_x \phi)^2 + V$$

$$= \frac{1}{2} (\partial_x \phi)^2 + \frac{\lambda}{4} (\phi^2 - F^2)^2 = \left[\frac{\partial_x \phi}{\sqrt{2}} \right]^2 + \left[\frac{\sqrt{\lambda}}{2} (\phi^2 - F^2) \right]^2$$

Remind that:

$$a^2 + b^2 \geqslant 2 |ab|$$

Ergo:

$$E = \int_{-\infty}^{\infty} dx \, \mathcal{E}(x) = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_x \phi)^2 + V \right]$$

$$\geqslant \int_{-\infty}^{\infty} dx \; 2 \cdot \frac{1}{\sqrt{2}} \frac{\sqrt{\lambda}}{2} \left| \partial_x \phi \, (\phi^2 - F^2) \right|$$

$$\geqslant \sqrt{\frac{\lambda}{2}} \left| \int_{-\infty}^{\infty} dx \; \partial_x \phi \, (\phi^2 - F^2) \right| = \sqrt{\frac{\lambda}{2}} \left| \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi \, (\phi^2 - F^2) \right| =$$

$$= \sqrt{\frac{\lambda}{2}} \left[\frac{1}{3} \phi^3 - F^2 \phi \right]_{\phi(-\infty)}^{\phi(+\infty)} = \sqrt{\frac{\lambda}{2}} \left| \frac{\phi}{3} (\phi^2 \right.$$

If case:

$\phi(-\infty) = \phi(+\infty) = F$

$E \geq \mathcal{O}$

trivial constrain.

But, if you check $\phi(-\infty) = -F$

$\phi(+\infty) = F$ we have

you get:

$$E \geq \sqrt{\frac{\lambda}{2}} \left| \left[\cdot 2 \cdot \left(\frac{F^3}{3} - F^3 \right) \right] \right| = \sqrt{2\lambda} \cdot F^3 \cdot \left(\frac{2}{3} \right)$$

$$= \frac{2}{3} F^3 \sqrt{2\lambda} = M_{KINK}$$

Recall:

$M_{KINK} = \frac{2}{3} F^2 m$

$m = \sqrt{2\lambda}\, F$

Recall that

This holds a general procedure...

$E \geq \text{const} \cdot Q$

$E = \text{const}\ (Q = 1)$ for a soliton.

$E \geq M_{KINK} \cdot |Q|$.

Namely:

$$E \geq \sqrt{\frac{\lambda}{2}} \cdot \left[\phi^2 \cdot \frac{2}{3}\phi\right]_{\phi(-\infty)}^{\phi(+\infty)} =$$

$$= \sqrt{\frac{\lambda}{2}}\, F^2 \cdot \frac{2}{3} \left(\phi(+\infty) - \phi(-\infty)\right)$$

$$= \sqrt{\frac{\lambda}{2}}\, \frac{2}{3} F^2 \left|\frac{\phi(+\infty) - \phi(-\infty)}{2F}\right| \cdot 2F$$

$$\boxed{E \geq \sqrt{\frac{\lambda}{2}}\, \frac{4}{3} F^3 \cdot |Q|}$$

$$\boxed{E \geq M_{kink} \cdot |Q|}$$

Program.

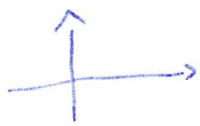
Fr. 10 → KINK + Q and Bog

Fr. 17
- W general
- wavy potential
- Definition solitary wave and solitons
- Quantisation → short description
- Derrick's theorem

W QM as a ft

$L = \frac{1}{2}\dot{x}^2 - V(x)$

$x = x(t)$



This is a point

(11-)?

Fr. 25 W as instanton ... in detail.

Fr. 31 O(3) in $\underline{1+2}$ as class. theory ; → go one dim more...

7/6 O(3) in 1+1 as QFT

15/6 Vortex and its gauge version in 2+1!!

25/6 Monopole / Dirac + SU(3) in 3+1...