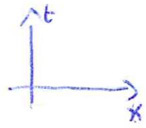
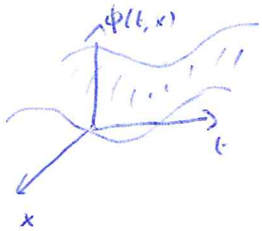
FT with one scalar field : RECALL
in 1+1 dim.

$\phi = \phi(t,x) : \mathbb{R}^2 \mapsto \mathbb{R}.$





$\partial_\mu = (\quad \text{with}) \qquad \mu = 1, 2$

$$\begin{cases} \partial_0 = \partial_t \\ \partial_1 = \partial_x \end{cases}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - V$$

Dimensions: $[\mathcal{L}] = \text{Energy}^2$ ($S = \int dt\, dx\, \mathcal{L}$ is dimensionless)

$[\phi] = \text{Energy}^0 = 1$: Dimensionless.

(in 4d = (1+3)d $[\phi]$ has dim. energy)

E.o.m.:

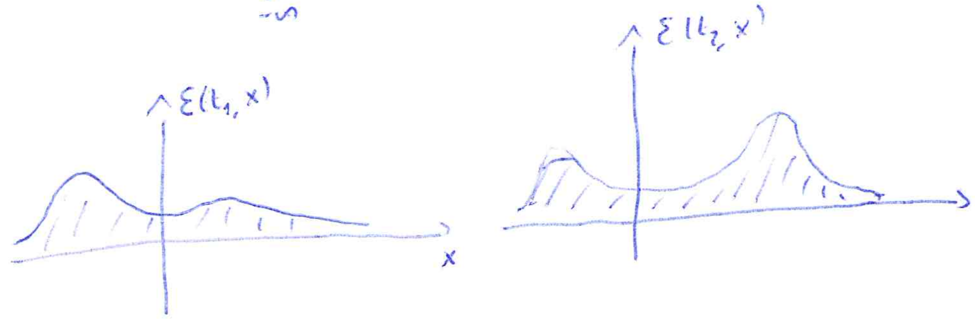
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \longmapsto \boxed{(\partial_t^2 - \partial_x^2)\phi = -\frac{\partial V}{\partial \phi}}$$

Energy density

$$\mathcal{E} = \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + V(\phi)$$

If $\phi(t,x)$ solves the e.o.m. (and $\mathcal{E}$ vanishes for $x \mapsto \pm\infty$):

$$E[\phi] = \int_{-\infty}^{\infty} dx \; \mathcal{E}(\phi(t,x)) = \text{const.} \quad \forall t \qquad \left(\frac{dE}{dt} = 0\right)$$



the area is the total energy and is a constant, provided that $\phi(t,x)$ is a solution of the e.o.m. and is such that $\mathcal{E}$ vanishes for $\phi(x \mapsto \pm\infty)$

How to prove it?
A constant quantity such as the energy is such if there is a corresponding conserved current.

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \quad , \quad \partial_\mu T^{\mu\nu} = 0$$

$\nu = 0$

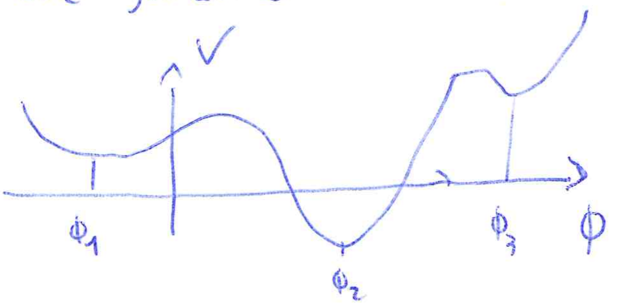
$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_1 T^{10} = \partial_0 \mathcal{E} + \partial_1 T^{10} = 0 \qquad \left(P = -(\partial_t \phi)(\partial_x \phi)\right)$$

$$E = \int_{-\infty}^{\infty} dx \; \mathcal{E} \; ; \quad \frac{dE}{dt} = \int_{-\infty}^{\infty} dx \; \partial_t \mathcal{E} = -\int_{-\infty}^{\infty} dx \,(\partial_x T^{10}) = -\left[T^{10}(x = +\infty) - T^{10}(x = -\infty)\right]$$

$$= 0$$

Trivial remark:

how does one guarantee that $\mathcal{E}(x,t)$ vanishes for $x \mapsto \pm\infty$?



$\phi(x \mapsto \pm\infty, t) = \phi_1 = \text{const} \ldots$ $\quad \left(\frac{\partial V}{\partial \phi} = 0 \right)$

$(\phi_2, \phi_3, \ldots)$

ϕ must tend to a minimum for $x \mapsto \pm\infty \ldots$

($\phi(x \mapsto -\infty, t)$ is not necess. equal to $\phi(x \mapsto +\infty, t)$, if more minima are present).

Easiest choice: free FT.

$V(\phi) = \frac{1}{2} m^2 \phi^2$ $\qquad [m] = \text{Energy}$

E.o.m:

$$\boxed{(\partial_t^2 - \partial_x^2)\phi = -m^2 \phi}$$

• Let us search for solution which depend on t only:

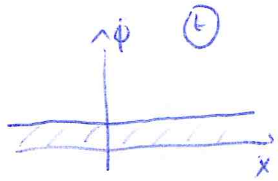
$\phi = \phi(t)$

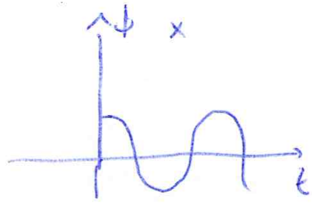
$$(\partial_t^2 + m^2)\phi = 0$$

$\phi = A\cos(mt)$ is a solution.

$$\mathcal{E} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2} m^2 \phi^2 = \frac{1}{2} A^2 m^2 \sin^2(mt) + \frac{1}{2} A^2 m^2 \cos^2(mt) = \frac{A^2 m^2}{2} = \text{const.}$$

$E = \infty$





$\mathcal{E}$ does not vanish for $x \mapsto \pm\infty$ ....

N.b: there are instants for which $\phi = 0$ ... $\left(t = \frac{1}{m}\frac{\pi}{2}\right)$.

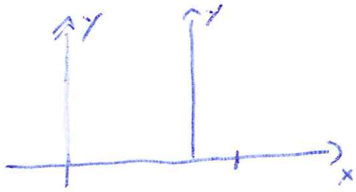
still, $\mathcal{E} \neq 0$, because $\partial_t \phi \neq 0$...

Nothing new under the sun: this solution is a plane wave...

$\phi(x \mapsto \infty, t) \neq 0$ in general.

$\phi(t,x) = \phi(t) = A\cos(mt)$ is a solution of e.o.m., but one which costs an ∞ amount of energy to make.

How to make other solutions? Answer: via "boost".



$x = x' + vt$

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \qquad \boxed{\gamma = \frac{1}{\sqrt{1-v^2}}}$$

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

$$A\cos(mt') = A\cos(m(\gamma t - \gamma v x)) = A\cos(\omega t - kx),$$

where

$$\begin{cases} \omega = \gamma m = \dfrac{m}{\sqrt{1-v^2}} \\ k = m\gamma \cdot v \end{cases}$$

$$\omega = \sqrt{k^2 + m^2} = \sqrt{m^2\gamma^2 v^2 + m^2} = m^2\sqrt{\frac{v^2}{1-v^2} + 1} = \sqrt{m^2\,\frac{v^2 + 1 - v^2}{1-v^2}} = \gamma m$$

q.e.d.

$A\cos(\omega t - kx)$ is a sol. of the eom.

- $(\partial_t^2 - \partial_x^2)\phi(t,x) = -A\underbrace{(\omega^2 - k^2)}_{= m^2}\cos(\omega t - kx) = -m^2 A\cos(\omega t - kx)$ ✓

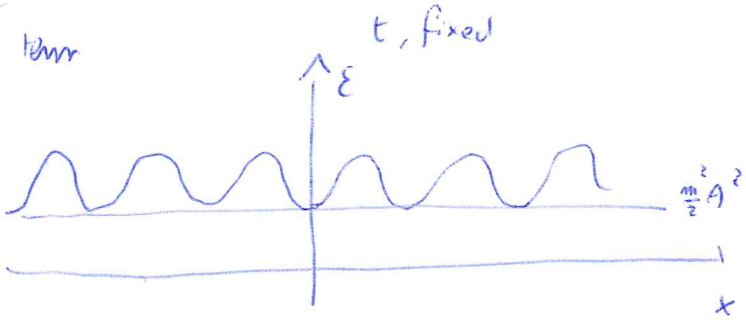
it is a solution of the e.o.m., but:

- $\mathcal{E} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 = \frac{1}{2}A^2\omega^2\sin^2(\ldots) + \frac{1}{2}k^2A^2\sin^2(\ldots) + \frac{1}{2}m^2A^2\cos^2(\ldots)$

$= \frac{A^2}{2}(\omega^2 + k^2)\sin^2(\omega t - kx) + \frac{m^2}{2}A^2\cos^2(\omega t - kx)$

$(\omega^2 = k^2 + m^2)$

$= \underbrace{A^2k^2\sin^2(\omega t - kx)}_{\text{space-time dependent term}} + \underbrace{\frac{m^2}{2}A^2}_{\text{constant term}}$



Also here the total energy is ∞!

$\phi(t, x \mapsto \pm\infty) \neq 0$ (which is the minimum of the potential).

How to build a finite energy solution? → Wave packet!

(Actually, one should also use wave packets in FT and QFT; the problem is that they are more diffic. to use...)

$$\varphi_{WP}(t,x) = \int_{-\infty}^{\infty} f(k)\, \phi_k(t,x)\, dk$$

$$\boxed{\phi_k(t,x) = \cos(\omega t - kx)}$$

In view of the fact that each ϕ_k is a solution of the e.o.m., it follows that $\varphi(t,x)$ is also such.

Obviously, for each choice of $f(k)$ one has a different solution.

Dispersion:

$$\varphi(t,x) = f(\omega t - kx) = f(u)$$

Is it a solution for each choice of f?

$$\partial_t f = \frac{\partial f}{\partial u}\omega \qquad \partial_x f = -\frac{\partial f}{\partial u}\cdot k$$

$$\partial_t^2 f = \frac{\partial^2 f}{\partial u^2}\omega^2 \qquad \partial_x^2 f = +\frac{\partial^2 f}{\partial u^2}k^2$$

$$(\partial_t^2 f - \partial_x^2 f) = \frac{\partial^2 f}{\partial u^2}(\omega^2 - k^2) = -m^2 f^2$$

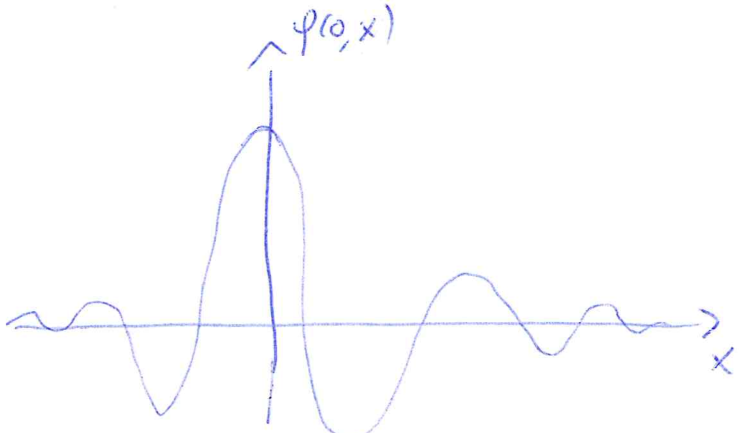
→ each choice of $f(u)$ is not a solution for $m>0$... only if $m=0$ it is like that, otherwise we get a plane wave only...

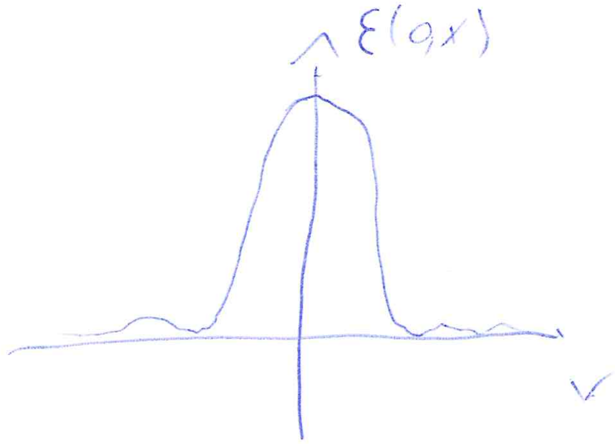
Let us do an "easy choice":

$$\varphi(t,x) = B\int_{-\Delta k/2}^{\Delta k/2} \cos(\omega t - kx)\,dk$$

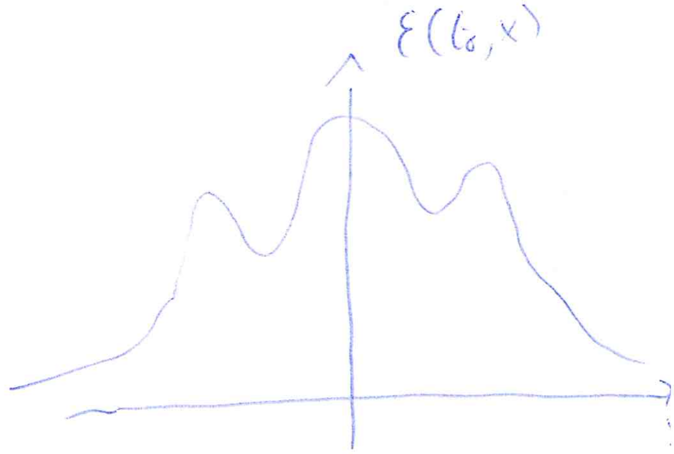
(Not trivial, because $\omega = \sqrt{k^2+m^2}$).

$\varphi(t, x \mapsto \pm\infty) = 0$





One observes: spreading of the w.p. for increasing time



Conclusion: the w.p. spreads... it is in this sense not a stable bulk of energy

(For $m=0$ -> same velocity... this is because all elements have the same velocity ($v=1$)).

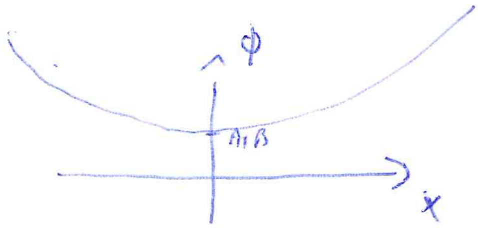
One can also boost the w.p., but we still have a w.p. moving and spreading.

Let us go back to the eom:

$$(\partial_t^2 - \partial_x^2)\,\phi(t,x) = -m^2 \phi(t,x).$$

Let us search for a solution of the type $\phi = \phi(x)$. space-only...

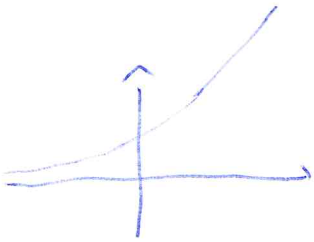
$$\partial_x^2 \phi(x) = m^2 \phi(x)$$



$$\phi(x) = A e^{mx} + B e^{-mx}$$

is a solution

$$\phi(x) = A e^{mx}$$



$$\mathcal{E} = \frac{1}{2}(\partial_x \phi)^2 + \frac{m^2}{2}\phi^2 = A^2 m^2 e^{2mx}$$

$$E = \infty \quad \forall t \quad (t\text{-independent}..)$$

We can indeed find solutions which depend from x only, but their energy is ∞...

it is ∞ in a $\neq$ way than the plane wave: here we really have a divergence of the field..

As before, we can construct other solutions by boost.

$$\phi(t,x) = Ae^{mx'} + Be^{-mx'} = Ae^{m(-\gamma vt+\gamma x)} + Be^{-m(-\gamma vt+\gamma x)}$$

One can verify that it is a solution of e.o.m... still, one with ∞ energy.

Here we cannot obtain finite energy solutions by building a wave packet...

we would get solutions which still have ∞ energy.

$$V(\phi) = \frac{m^2}{2}\phi^2 + \lambda\phi^4$$

- non-linear term in the e.o.m. (sum of solutions is not a solution

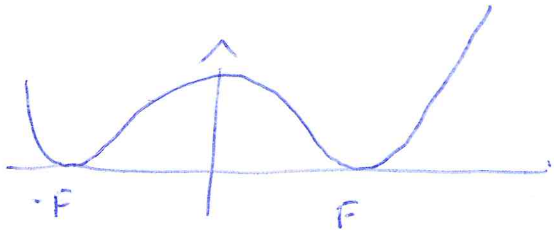
- still one minimum only: "same topology"

- similar discussion: no solution of the type $\phi = \phi(x)$.

- How: no general solution exist.

The KINK

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - F^2)^2$$



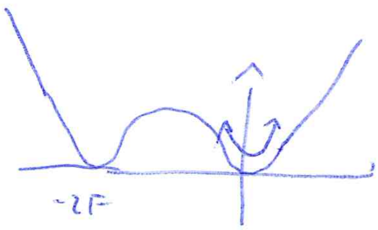
$[\lambda] = \text{Energy}^2$ (very different from 1+3 world...).

$[F] = \text{Energy}^0$.

$$(\partial_t^2 - \partial_x^2)\phi = -\partial_\phi V = -\lambda\phi(\phi^2 - F^2)$$

- Digression 1:

if we do the shift $\phi \mapsto F + \phi$, we get:



$$V = \frac{\lambda}{4}(\cancel{F^2} + \phi^2 + 2F\phi - \cancel{F^2})^2 =$$

$$= \frac{\lambda}{4}(\phi^4 + 4F^2\phi^2 + 4F\phi^3).$$

$\boxed{m^2 = 2\lambda F^2}$ -> energy of the fluctuation... if we would do a perturbation expansion starting from the vacuum...

- Digression 2:

$$V = c\phi^n$$

$[c] = \text{Energy}^2$

-> it is renormalizable... each interaction is such in 1+1 d.

Back to the e.o.m. where we search a solution of the type:

$\phi = \phi(x)$
↳ space only

$$\boxed{+\partial_x^2 \phi = +\lambda \phi (\phi^2 - F^2)}$$

Now, I just "guess" a function and let us see if it is a solution:

$$\phi(x) = F \tanh\left[\alpha x\right] \qquad (m^2 = 2\lambda F^2)$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \boxed{\lim_{x \mapsto \pm\infty} \tanh x = \pm 1}$$

$$\partial_x \tanh x = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{1}{ch^2 x}$$

$$\partial_x \left(\frac{1}{ch^2 x}\right) = -\frac{2\, ch x\, sh x}{ch^3 x} = -\frac{\tanh x}{ch^2 x}$$

Let us check:

$$\partial_x \phi = F \alpha \frac{1}{ch^2(\alpha x)}$$

$$\partial_x^2 \phi = -2F \alpha^2 \frac{\tanh(\alpha x)}{ch^2(\alpha x)}$$

Plug in:

$$-2F\alpha^2 \frac{\tanh(\alpha x)}{\mathrm{ch}^2(\alpha x)} = \lambda \cdot F \cdot \tanh(\alpha x) \cdot F^2 \left(\tanh^2(Fx) - 1\right)$$

$$= -\lambda F^3 \tanh(\alpha x) \frac{1}{\mathrm{ch}^2(Fx)}$$

$$2\alpha^2 = \lambda F^2$$

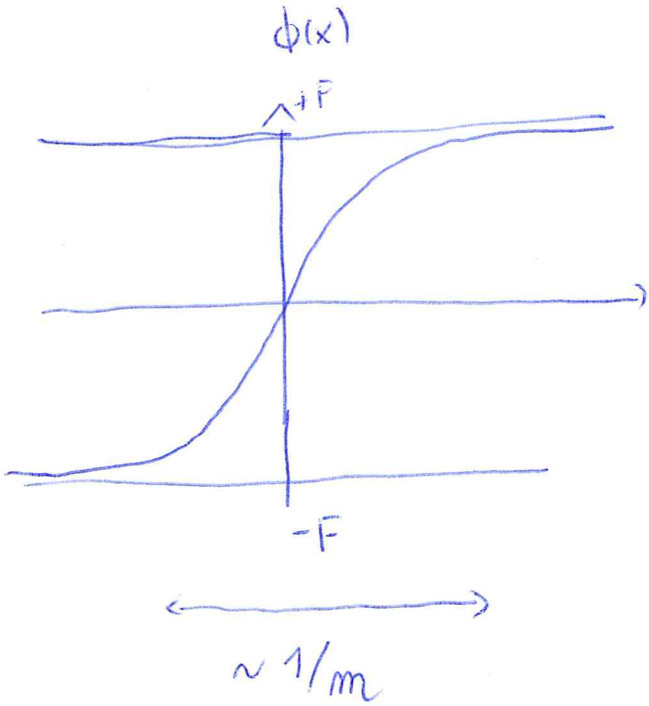
$$\boxed{\alpha^2 = \frac{1}{2}\lambda F^2} \qquad \alpha = \frac{1}{\sqrt{2}}\sqrt{\lambda}\, F. \qquad m = \sqrt{2\lambda}\, F^2$$

$$\frac{m}{2} = \frac{\sqrt{\lambda}}{\sqrt{2}} F$$

$$\alpha = \frac{1}{\sqrt{2}} \lambda F$$

Ergo, the solution is also:

$$\boxed{\phi(x) = F \tanh\left(\frac{m}{2} x\right)}$$



The energy E is finite, as we will show...

Basic considerations

$V \sim \phi^4$, but now due to the nontrivial form (and, in particular, to the presence of two minima) a new kind of solution was obtained.

- Non-trivial solution (and nonperturbative as well) has been found..

- stable: no spread with time

- $E(M) = \int_{-\infty}^{\infty} \varepsilon(t,x)\,dx \propto \frac{m^3}{\lambda}$ $1/\lambda$ .. λ very small, energy very large...

  This solution is not obt. in perturbation theory.

- By translation and boost: new solutions can be obtained

Problem: we determined the solution by guessing..

Unfortunately, this is the only known way to get topological solutions,