Instantons in YM theory (and in QCD)

SU(2) ($N_c = 2$ for simplicity)

Weyl gauge: $A_0^a = 0$; $A_i^a(t, \vec{x})$ determines the field.

This is an interesting gauge because the action takes the form

$$S = -\int d^4x \, \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} = -\int d^4x \, \frac{1}{2} \left(\vec{E}^{a\,2} - \vec{B}^{a\,2} \right)$$

where:

$$\begin{cases} E_i^a(t,\vec{x}) = \dfrac{\partial}{\partial t} A_i^a(t,\vec{x}) \\[2ex] B_i^a(t,\vec{x}) = \dfrac{1}{2} \varepsilon_{ijk} \left(\partial_j A_k^a - \partial_k A_j^a + \varepsilon^{abc} A_j^b A_k^c \right) \end{cases}$$

This means that we can rewrite S as:

$$S = -\int dt \left[\underbrace{\frac{1}{2} \int d^3x \left(\frac{\partial A_i^a}{\partial t} \right)^2}_{E_i^a \;\to\; \text{kinetic term}} - \underbrace{\frac{1}{2} \int d^3x \left(B_i^a \right)^2}_{\text{"potential" term}} \right]$$

Of course, this should be seen with care because the potential term also contains derivative... but these are actually spatial derivatives, not w.r.t. time.

e.o.m.: RECALL

N_c "in general"

$$\boxed{D_\mu F^{\mu\nu} = 0}$$ very compact expression $D_\mu = \partial_\mu - ig[A_\mu, \cdot]$, $F^{\mu\nu} = F^{a,\mu\nu} t^a$.

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] = 0$$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]) - ig[A_\mu, F^{\mu\nu}] = 0$$

$$\boxed{\partial_\mu F^{a,\mu\nu} + g f^{abc} A_\mu^b F^{c,\mu\nu} = 0}$$

$N_c = 1 \rightarrow$ trivial limit

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \rightarrow \text{eom of the gauge potential..}$$

$N_c = 2$

$$\partial_\mu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\varepsilon^{abc} A_\mu^b A_\nu^c) + g\varepsilon^{abc} A_\mu^b (\partial^\mu A^{c,\nu} - \partial^\nu A^{c,\mu} + g\varepsilon^{c\alpha\beta} A^{\alpha,\mu} A^{\beta,\nu})$$

$= 0$

↳ extremely complicated eq. ...

Nevertheless, it comes from a extremely symmetric theory.

Topological charge: conserved quantity which is only related to the general property of fields and space time and not on the chosen Lagrangian.

In this 1+3 world with (gauge) field the topological charge is

$$Q_T = \int \frac{d^4x\, g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu}$$

where:

$$\tilde{F}^{a,\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

The quantity $F\tilde{F}$ can be expressed as a total divergence:

$$\frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} = \partial_\mu K^\mu ; \quad \boxed{K^\mu = \frac{g^2}{16\pi^2} \varepsilon^{\mu\alpha\beta\gamma} \left(A^a_\alpha \partial_\beta A^a_\gamma + \frac{g}{3} \varepsilon^{abc} A^a_\alpha A^b_\beta A^c_\gamma \right)}$$

$F\tilde{F}$ is then a full divergence!

Why should it play a role at all?

$F\tilde{F}$ is a full divergence.

Does it mean that I can simply make the replacement

$$\mathcal{L}_{YM} \mapsto \mathcal{L}_{YM} + \kappa F^{a,\mu\nu}\tilde{F}^{a}_{\mu\nu}$$

and nothing happens?

This is indeed at first sight the case... one can namely add a full derivative to a Lag, and no physical consequence emerges...

How, ACHTUNG: this is the case only if trivial fluctuations are considered. Namely:

$$\int d^4x\, \partial_\mu K^\mu = \int d^4x\, (\partial_0 K^0 - \partial_i K^i) =$$

$$= \int d^4x\, \partial_0 K^0 - \int d^4x\, \partial_i K^i = \int dt \frac{\partial}{\partial t}\left(\int d^3x\, K^0\right) - \int dt \int d^3x\, \partial_i K^i =$$

$$= \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} N_{CS} - \underbrace{\int dt \int_{S_\infty} \vec{J}\vec{n}\, dS}_{=0} = \underbrace{N_{CS}(\infty) - N_{CS}(-\infty)}_{\text{di}}$$

$$\boxed{N_{CS}^{(t)} = \int d^3x\, K^0}$$

Chern-Simons number $N_{CS}(t)$.

James Simons: first mathematician, then hedge fund... → USING MATHS! employment of mathematicians and physicists, 79th richest man in the world

Philanthropy!

Now, if you consider that the only vacuum is the trivial configuration

$A_\mu = 0$, then obviously

$$N_{CS}(t) = \int d^3x\, K_0 = \int d^3x\, \varepsilon^{ijk}\left(A_i^a \partial_j A_k^a + \frac{g}{3}\varepsilon^{abc} A_i^a A_j^b A_k^c\right) = 0.$$

Still, we might have other vacua in a YM theory...

In PRINCIPLE, each pure gauge configuration

$$A_\mu^a = -\frac{i}{g} U \partial_\mu U^\dagger \quad \text{is such.}$$

Let us now consider a time-independent gauge configuration

$$\boxed{A_i^a(\vec{x}) = -U(\vec{x})\, \partial_i U^\dagger(\vec{x})}$$

one can show that

$$N_{CS} = N_W = \frac{1}{24\pi^2} \int d^3x\, \varepsilon^{ijk} (U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U)$$

We do not need to understand the details of it... but there is indeed an important point.

What does a pure gauge?

It can be seen as a Mapping

$$U(\vec{x}): R_3 \mapsto SU(2) \sim SO(3) = S_3 = \text{sphere!}$$

However, if in addition we require that $U(\vec{x} \mapsto \infty) = U_0$ (Fixed!!!)

we wrap the space R_3 and make a sphere out of it.

We then have the mapping

$$U(\vec{x}): S_3 \mapsto S_3$$

which is non trivial (just as $\bigcirc \mapsto \bigcirc$, remember?

or ☺ → ☺ $O(3)$ model)

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By making the choice

$$U = e^{i\vec{r}\cdot\vec{\tau}/r\, P(r)}$$

one get

$$N_{CS} = N_{CS}(t) = N_W = \frac{2}{\pi}\int dr \frac{dP}{dr}\sin^2 P = \frac{1}{\pi}\left[P(r) - \frac{\sin(2P(r))}{2}\right]_0^\infty =$$

↳ No t dependence here

integer

if $P(r)$ must be such

$P(0) = m_1 \pi$

$P(\infty) = m_2 \pi$

This is necessary for $U(\vec{r})$ being defined in 0 and ∞!!!!

This means:

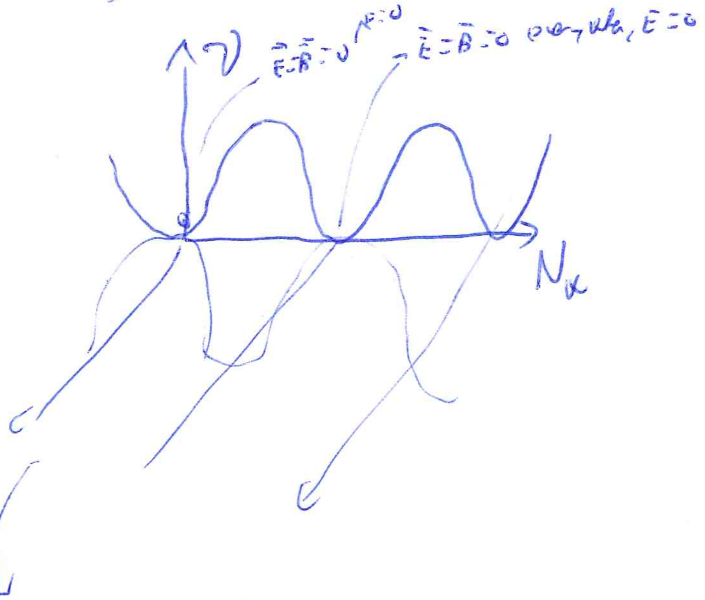
each choice of $U(\vec{r})$ with $\boxed{A_i(\vec{r}) = -U(\vec{r})\,\partial_i U(\vec{r})}$ is a possible vacuum.

Still, not all the vacua are identical, but come in different classes, associated to their winding number.

$A_i = 0 \qquad N_w = 0$

but in general N_w is an integer.

Intuitively, the potential $V = \frac{1}{2}\int d^3x\, \vec{B}^{a\,2}$ has minima for N_w being an integer. (Minima $= 0$, pure gauge...)

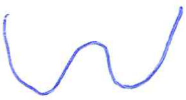


other coordinates...

This means however that the YM vacuum has many different minima, according to which N_w is realised.

But here the instantons come into game.

Recall that: Double-well, QM



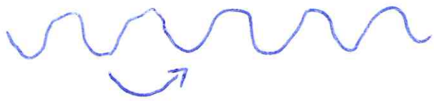
$T \propto e^{-S_E}$

minimum of the euclidean action

$t = -i\tau$

solve the euclidean eom.

YM



$\mathcal{L}_{YM} \mapsto \mathcal{L}_{YM}^{E}$

study solution of the euclidean eom.

The nontrivial ones with finite action describe the tunnelling from one vacuum to the other!!!!

$\mathcal{L}_{YM} \mapsto \mathcal{L}_{YM}^{E}$

$F^a_{\mu\nu} \mapsto F^a_{E\mu\nu}$, (we omit E, but we should note that we are now in Euclidean space).

$$0 \leq \int d^4x_E \left(F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu}\right)^2 = \int \underset{\text{omit } E}{d^4x} \left(F^{a\,2}_{\mu\nu} + \tilde{F}^{a\,2}_{\mu\nu} - 2 F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}\right) =$$

$$= \int d^4x \left(2F^{a\,2}_{\mu\nu} - 2 \underbrace{F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}}_{Q_T}\right) = 8 S_E - 64\pi^2 \frac{Q_T}{g^2}$$

We then obtain:

$$S \geq \frac{8\pi^2}{g^2} Q_T = \frac{8\pi^2}{g^2} \quad \text{if } Q_T = 1.$$

The minimum of the action is obtained for

$$S = \frac{8\pi^2}{g^2} \longleftrightarrow F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$$

Being $D_\mu \tilde{F}^{\mu\nu} = 0$ always fullfilled, we also realise that the eom $D_\mu F^{\mu\nu} = 0$ is fullfilled as well.

The Euclidean action is then $\frac{8\pi^2}{g^2}$.

The tunnelling probability between the two vacua is:

$$T \sim e^{-\frac{8\pi^2}{g^2}} = e^{-\frac{2\pi}{\alpha_s}}$$

$\hookrightarrow$ Typical form... in fact, in all models we have always found $\sim \frac{1}{g^2}$...

g small $\mapsto$ T small as well.

The field A_μ^a which fullfills the requirements is:

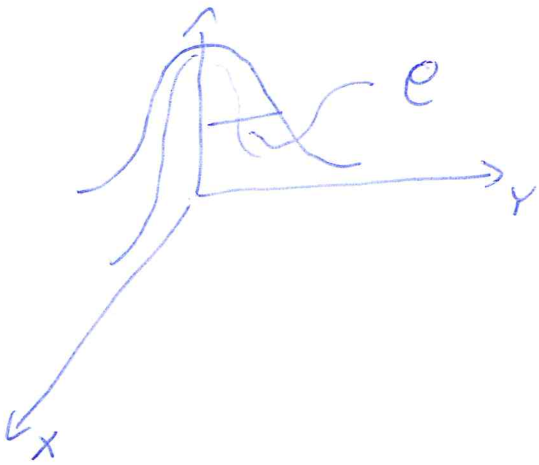
$$A_\mu^a = \bar{\eta}_{\mu\nu}^a \frac{2\rho^2}{x_E^2(x_E^2+\rho^2)} \qquad x_E^2 = \vec{x}^2 + y^2$$

't Hooft symbols

$$\bar{\eta}_{\mu\nu}^a = \varepsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{\mu 4}$$

$$F_{\mu\nu}^a F^{a,\mu\nu} = \frac{192\rho^4}{(x_E^2+\rho^2)^4}$$

This is a "bump" in $x_E = 0$



Still, ρ is not fixed. The E-YM theory is still invariant under dilatation.

This configuration interpolates between $N_{CS}=0$ and $N_{CS}=1$.

obviously, we can:

- change the center (4 coordinates)
- rotation in colorspace ($4N_c - 5$)
- dimension ℓ

Remind the gluon condensate?

$$\frac{1}{32\pi^2}\langle F^a_{\mu\nu}F^{\mu\nu,a}\rangle = \frac{1}{16\pi^2}\langle \vec{B}^{a\,2} - \vec{E}^{a\,2}\rangle \sim \underbrace{(200\ \mathrm{MeV})^4}_{\Lambda_{YM}^4}$$

The interesting fact is that the instantons in the YM vacuum can explain the emergence of such condensate.

Note that at <u>each order in perturbation theory</u>

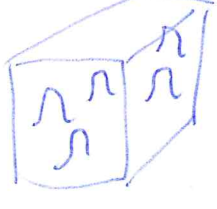
$$\langle \vec{E}^{a\,2}\rangle = \langle \vec{B}^{a\,2}\rangle \to \langle FF\rangle = 0 .$$

So, this is a non-perturbative phenomenon.

The instantons can explain it because they are such

$$\begin{cases} \vec{E}^a = i\vec{B}^a \to \\ \langle \vec{B}^{a\,2} - \vec{E}^{a\,2}\rangle = 2\langle \vec{B}^{a\,2}\rangle \neq 0 !!! \end{cases}$$

How can we estimate it?



V_4 = four-dimensional volumes...

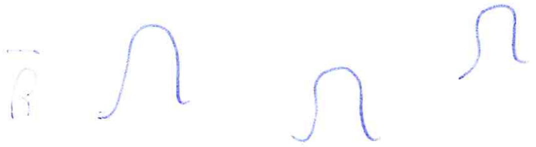
This is a measured quantity with instantons (numerically)

$$\frac{1}{32\pi^2}\underbrace{\langle F_{\mu\nu}^2\rangle}_{\text{density}} = N\cdot\left(4\,\frac{8\pi^2}{g^2}\right)\Big/ V_4$$

N: Number of instantons in V_4

Ergo, the gluon condensate is

$$\frac{1}{32\pi^2}\langle F^a_{\mu\nu}F^{a,\mu\nu}\rangle \simeq \frac{N}{V_4} \simeq \frac{1}{\bar\rho^4}$$



This picture is valid only for dilute instanton gas!

Here we have also to invoke dimensional transmutation.

One particular dimension is picked up!

$$\frac{N}{V_4} \simeq \frac{1}{\Lambda_{YM}^4}$$

There is one instanton in each cube with size $1/\Lambda_{YM}^4$

Thus, instantons + dim. trans. explain the emergence of $\langle FF\rangle$.

So, the fact that the dilaton potential has a minimum for $G_0 \neq 0$ is due to the instantons.

of course, all this is "heuristic"..

One should, in a complete framework, calculate the fluctuation...

this is then technically very difficult.

$$A_\mu = A_\mu^I + a_\mu$$

↳ fluctuations

hard job.

In the end, one gets the contribution of an ensemble of instantons and antiinstantons.

→ Diakonov's papers.

(story of Diakonov...)

$d(\rho)\,d\rho$ ≡ probability that an instanton has a dimension between ρ and $d\rho$ ...

First calculation gives something like

$$d_0(\rho) \sim \frac{1}{\rho^5}\,\rho^{\frac{11}{3}N_c}$$ → it diverges for large ρ!!!

problem of overlapping instantons.

The regularisation takes place as

$$d(\rho) = d_e(\rho)\, e^{-\#\, \rho^2 \Lambda_{YM}^2}$$

One can show by variational calculations that it works.

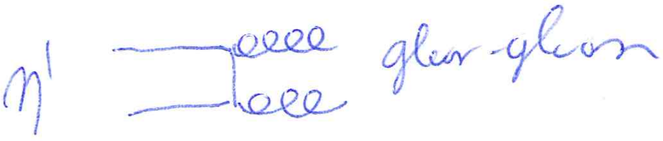
Also on lattice one can see that these instanton configurations appear.

The dilute instanton gas can be justified a posteriori.

Other interesting effects with instantons:

* $m_{\eta'}^2 \sim \langle Q_T^2 \rangle$

Intuitive... $\eta' \sim F_{\mu\nu}\tilde{F}^{\mu\nu}$ couples to $\eta' \stackrel{\sim}{=} \sqrt{\frac{1}{3}}\,(\bar{u}u + \bar{d}d + \bar{s}s)$



This is also the so-called chiral anomaly!

* $\theta F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}$ can be added to the Lag of QCD. θ is a free parameter and, because of the existence of instantons, is not trivial even if this term is a full derivative.

Measurement: $\theta \simeq 0$. Why? Strong CP problem!

Implication and model of hadrons (eLSM):

$\mathcal{L} = \mathcal{L}_{dil} + \mathcal{L}_{mesons} = T - V$

$$V = \text{Kin term} + \underbrace{\frac{1}{4}\frac{m_G^2}{\Lambda_G^2}G^4\left(\ln\frac{G}{\Lambda_G} - \frac{1}{4}\right)}_{\to \text{ instantons}} - \frac{a}{2}G^2(\sigma^2+\vec{\pi}^2) + \lambda(\sigma^2+\vec{\pi}^2)^2$$

a, λ dimensionless!

The only dimension is in Λ_G ($\sim N_c^2 \Lambda_{YM}$!!!)

G condenses: $G \simeq G_0 \sim \Lambda_G$

Then, for σ and $\vec{\pi}$ we get

$$V = -\frac{a}{2}G_0^2(\sigma^2+\vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 \simeq -\frac{a}{2}G_0^2\sigma^2 + \frac{\lambda}{4}\sigma^4$$

$\rightsquigarrow$ σ condenses as well: $\frac{\partial V}{\partial \sigma} = -a G_0^2\sigma + \lambda\sigma^3 \to \sigma_0 = \boxed{\phi = \sqrt{\frac{a}{\lambda}}\,G_0}$

ϕ is the chiral condensate, that is proportional to Λ_G... that is G_0 instantons!!!!

(In addition $\to \eta'$ mass, as explained before... there is an explicit term which takes care of that).

$\to$ This is the eLSM that we study here and that can explain hadron masses and decays