$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu} \right]$$

→ dimensionless coupling constant

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_0 \left[A_\mu, A_\nu \right]$$

↳ I call it now g_0 (e is for the electric charge, which however here is never present)

$$A_\mu = A_\mu^a t^a \qquad a = 1, \dots, N_c^2 - 1$$

↳ in general, for a nr of colours equal to N_c

[$N_c = 1 \to$ eQED
$N_c = 2 \to$ gauge group of SM (here broken by the Higgs)]

$N_c = 3 \mapsto$ QCD

$N_c = 4, 5, \dots$ speculative (extension of the SM) (SU(5) was a hot candidate → political → ruin of physics?)

anyhow: useful "trick" to simplify owns life

t^a are $N_c \times N_c$ matrices...

they are the generators of the SU(N) group.

Although these are most probably known concept, it is good to review some basic properties of this group.

Note:
- only gluons here... no quarks
- self interactions are present for $N_c = 2$,

$SU(N)$: a subgroup of $N \times N$ complex matrices

$U \in SU(N)$ if: $U^\dagger U = U U^\dagger = 1$

$\det U = 1$

As a solution we can write down

$$U = e^{i\theta_a t^a}$$

where: $t_a^\dagger = t_a$ (Hermitian) $\mapsto U^\dagger U = 1$

$Tr[t_a] = 0 \rightarrow \det U = e^{i\theta_a Tr[t^a]} = e^0 = 1$

One makes the following choice:

$$\boxed{Tr[t_a t_b] = \frac{1}{2}\delta^{ab}}$$

$N = 2 \quad t_a = \frac{\gamma_a}{2}$ ⇝ Pauli

$N = 3 \quad t_a = \frac{\lambda_a}{2}$ ⇝ Gell-Mann

How many t_a? we have in general a $N \times N$ complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\rightarrow$ fixed by $2N^2$ real numbers.... but we have the condition $t_a^\dagger = t_a \rightarrow N^2$ eq.

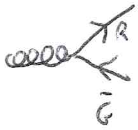
$Tr[t_a] = 0$ 1 eq

$\rightarrow 2N^2 - N^2 - 1 = N^2 - 1$

$a = 1, 2, \dots, N^2 - 1$

Intuitive understanding with gluons:

R, G, B



$3 \times 3 = 9$ combinations $\longrightarrow$ $-1 = 8$

$\downarrow$

$\bar{R}R + \bar{B}B + \bar{G}G$

The combinations are:

$R\bar{G}$, $\bar{R}G$,
$R\bar{B}$, $\bar{R}B$,
$G\bar{B}$, $\bar{G}B$,

$$\begin{cases} R\bar{R} - \bar{G}G \\ R\bar{R} + \bar{G}G - 2\bar{B}B \end{cases}$$

(The last one: $\bar{R}R + \bar{G}G + \bar{B}B$ is banned, because it is a "white" configuration...
it would be present with $U(N)$).

In YM theory one has a <u>local</u> gauge symmetry:

$$A_\mu \mapsto A'_\mu = U(x) A_\mu(x) U^\dagger(x) - \frac{i}{g_0} U(x) \partial_\mu U^\dagger(x)$$

where:

$$U(x) = e^{i\theta_a(x) t_a} \qquad a = 1, \dots, N_c^2 - 1$$

Limits: $N_c = 1$ $\qquad U(x) = e^{i\theta_a(x)}$

- $A_\mu \mapsto A'_\mu = A_\mu - \frac{i}{g_0} \partial_\mu \theta_a(x)$

- $\theta_a = \text{const}$

$$A_\mu \mapsto A'_\mu = U A_\mu U^\dagger$$

"usual", global" <u>color</u> transformation for a colored-gluon ...

if quarks transform as $q \mapsto Uq$, gluons transform as $U A_\mu U^\dagger$ ...

→ simple reshuffling of colors ...

One can easily prove:

$$G_{\mu\nu} \mapsto U(x)\, G_{\mu\nu}\, U^{\dagger}(x)$$

which means:

$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$ is an invariant object.

This is then proven for each N_c in a very general framework.

YM: Dimensional transmutation

Only one parameter: g_0. g_0 is dimensionless! It is a pure number.

Note, because of that the theory is invariant under dilatation transformations:

$$x^\mu \longmapsto \lambda^{-1} x^\mu = x'^\mu \qquad \lambda \in \mathbb{R}$$

Namely, if we also transform the fields as

$$A_\mu^a(x) \longmapsto A_\mu'^a(x') = \lambda A_\mu^a(x)$$

the Lagrangian transforms as:

$$\mathcal{L}\left(A_\mu^a(x)\right) \longmapsto \mathcal{L}'\left(A_\mu'^a(x')\right) = \lambda^4 \mathcal{L}\left(A_\mu^a(x)\right)$$

which implies that the action is invariant:

$$dS = d^4x\, \mathcal{L}\left(A_\mu^a(x)\right) = d^4x'\, \mathcal{L}\left(A_\mu'(x')\right) = dS'$$

This is expected: no dimension is in the theory, therefore it is all invariant under rescaling of space-time.

Note also that: if $A_\mu^a(x)$ is a solution of the e.o.m., then the function

$$A_\mu'^a(x) = \lambda A_\mu^a(\lambda x)$$

is also a solution of the eom.

This is the consequence of a symmetry: out of a solution of the eom, we can construct other solutions.

This property is what we have seen when writing down the dyon... dyons of each dimension exist at a classical level.

Even more formally, a symmetry is connected to a conserved current (Noether's theorem).

The "classically" "conserved" current of dilatation is:

$$J^\mu = x_\nu T^{\mu\nu} \quad , \quad T^{\mu\nu} = \frac{\partial \mathcal{L}_{YM}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - \eta^{\mu\nu} \mathcal{L}_{YM} + \text{"symm"}$$

Now, we realize that the situation is kind of paradoxical... if there is no scale, where does the proton mass come from? We know that the quarks can't do that ($m_u \simeq m_d \simeq 5\,\text{MeV}$, $M_p \simeq 1\,\text{GeV}$).

Even without invoking quarks, even $\mathcal{L}_{YM}$ ($N_c = 3$) generates bound state of gluons, the so-called glueballs. The mass of the lightest glueball in pure YM is about $1.5\,\text{GeV}$ (mass gap: from massless fields to massive glueballs).

But how can a number come in this theory?

This is one of the deepest issues in YM and, more in general, in QCD.

When considering the quantum version of the theory, one has to renormalize it, otherwise ∞'s pop out from everywhere.

The YM theory is indeed renormalizable (g_0 dimensionless).

When you renormalize a theory, you actually introduce a cutoff Λ, which is the maximal energy scale of that theory (that is, there is a minimal length $\ell_{min} \sim 1/\Lambda$).
Beyond Λ new physics will be present...

$\Lambda \lesssim M_{Planck} \simeq 10^{19}$ GeV

↳ merging of QFT and GRT, or one smaller scale...

(Λ could be also smaller, this is a kind of upper limit).

This is a physical interpretation of ren., in which the cutoff is a "real physical parameter" ⟼ Zee's book "QFT in a nutshell".

Here, for a ren. theory, one can prove that all low energy results do not depend on the high energy maximal scale Λ.

This is indeed the essence of ren. You do not need even to know the value of Λ... the dep. on Λ is of the type $1/\Lambda$, $1/\Lambda^2$, ... that is very small if Λ is large.

On the contrary, in a non-ren theory, Λ is a parameter which must be specified and which directly affects low-energy properties as well! (Example: NJL model).

Now, when doing renormalisation one obtains that

$$g_0 \xrightarrow{\text{Ren}} g(\mu)$$

$\hookrightarrow$ energy scale

$\partial_\mu J^\mu_D \neq 0$, but takes the form

$$\partial_\mu J^\mu_D = \frac{\beta(g)}{4g} G^a_{\mu\nu} G^{a,\mu\nu} \neq 0$$

where:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}$$

You see that if $g = g_0 = \text{const} \rightarrow \partial_\mu J^\mu_D = 0$, but in QFT this is not so.

At the 1-loop level

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = -b g^3 \qquad b = \frac{11 N_c}{48\pi^2}$$

This is an <u>anomaly</u>, that is a symmetry of the classical level of a QFT, which is broken at the quantum level.

(Discussion of the anomaly is important in QCD and in the SM)

It is called <u>trace anomaly</u>.

The solution of the flow equation

$$\mu \frac{\partial g}{\partial \mu} = -b g^3$$

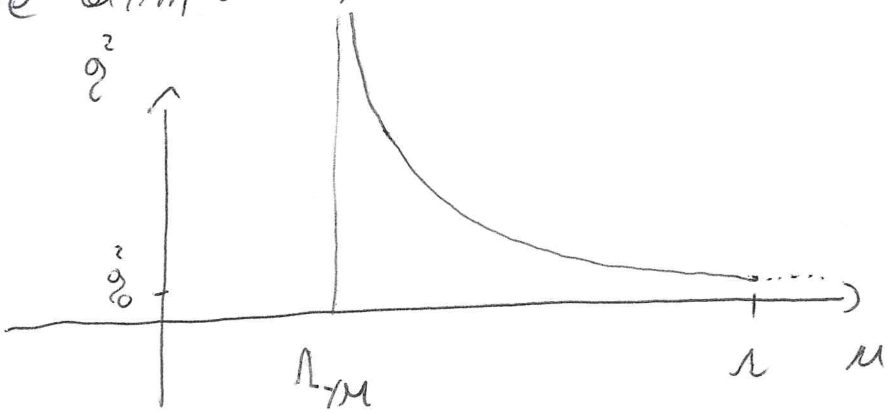
is

$$g^2(\mu) = \frac{g_0^2}{1 + 2 b g_0^2 \ln\left(\frac{\mu}{\Lambda}\right)}$$

g_0 "small number", is the value of the coupling constant when $\mu = \Lambda$

(Λ = high energy scale).

Note, we have asymptotic freedom...



$g(\mu)$ grows up for decreasing μ...

Indeed, there is a "pole"... for Λ_{YM} $g^2(\Lambda_{YM}) = \infty$.

Of course, this is "fake" in the sense that, when $g \mapsto \infty$, we cannot trust the 1-loop approximation.

But this is not the point... the point is that we have the emergence of a non-perturbative low-energy scale Λ_{YM},

$\hookrightarrow$ one can calculate it with other techniques... such as ERG...

$$1 + 2b g_0^2 \ln \frac{\mu}{\Lambda} = 0$$

$$\boxed{\Lambda_{YM} \equiv \text{"Landau pole"} = \Lambda \, e^{-\frac{1}{2b g_0^2}}}$$

(finite number) $\equiv$ (large number) * (small number) $\equiv$ " fin

Note: we cannot calculate Λ_{YM} because we don't know Λ and g_0.

But we can see how such a scale emerge.

In Nature (QCD)

$$\Lambda_{YM} \simeq 250 \text{ MeV} \,!$$

This is the scale out of which everything depends!

$f_\pi \simeq 100$ MeV

$M_p = 1$ GeV $\simeq 4\Lambda_{YM}$

$m^* \simeq \Lambda_{YM}$

A great achievement in QCD would be the calculation of M_p as function of Λ_{YM} <u>only</u>! (in the chiral limit, of course...).

In terms of Λ_{YM}

$$g^2(\mu) = \frac{1}{2b \ln\left(\frac{\mu}{\Lambda_{YM}}\right)}$$

Moreover, another important phenomenon takes place... one has a generation of a gluon condensate:

$$\langle T^{\mu}_{\mu} \rangle = -\left\langle \frac{11 N_C \alpha_s}{48\pi} G^{a}_{\mu\nu} G^{a,\mu\nu} \right\rangle \approx -\frac{11 N_C}{48} \Lambda^{4}_{YM}$$

Note that

$$\frac{1}{4} G^{a}_{\mu\nu} G^{a,\mu\nu} = \frac{1}{2} \Big(\underbrace{\vec{E}^{a\,2}}_{\text{chromoelectric field}} - \underbrace{\vec{B}^{a\,2}}_{\text{chromomagnetic field}} \Big)$$

In the Weyl gauge $(A^{a}_{0} = 0)$

$$\begin{cases} E^{a}_{i}(\vec{x}, t) = \dot{A}^{a}_{i}(\vec{x}, t) \\ B^{a}_{i}(\vec{x}, t) = \frac{1}{2} f_{ijk} \left(\partial_j A^{a}_{k} - \partial_k A^{a}_{j} + f^{abc} A^{b}_{j} A^{c}_{k} \right) \end{cases}$$

At a perturbative <u>level</u> $\langle \vec{E}^{a\,2} - \vec{B}^{a\,2} \rangle = 0$!

Gluons are indeed also confined; how can I describe these properties in an effective way?

One searches for a model which:

- reproduces the anomaly
- the emergence of a gluon condensate
- it is expressed only as function of a unique field G (glueball)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu G)^2 - V(G)$$

$$\partial_\mu J^\mu_D \sim G^a_{\mu\nu} G^{a,\mu\nu}$$

$$[G] = \text{Energy}$$

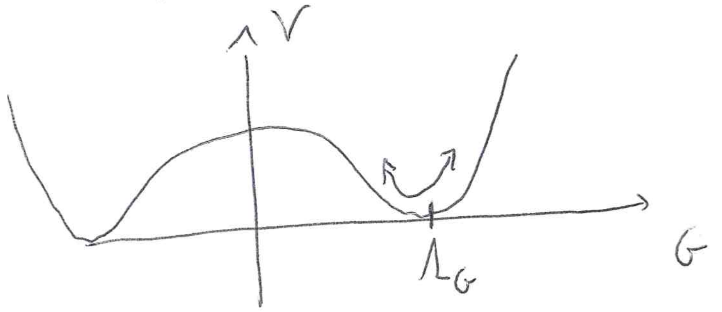
$$J^\mu_D = x_\nu T^{\mu\nu}$$

$$\partial_\mu J^\mu_D = \boxed{4V - G\partial_G V \propto G^4}$$

There is only one solution:

$$V = \frac{1}{4}\frac{m_G^2}{\Lambda_G^2}\left(G^4 \ln\frac{G}{\Lambda_G} - \frac{G^4}{4}\right)$$

The potential is such



$G_0 = \Lambda_G \to$ condensation

$G \mapsto G_0 + G$

↳ fluctuation with mass $m_G \to$ the glueball!!!!

Here the trace anomaly is reproduced by explicitly breaking it at the classical level of the composite Lag...

$$\boxed{\Lambda_G \sim N_c^2 \Lambda_{YM}}$$

One obtains

$$\partial_\mu J_D^\mu = -\frac{1}{4}\frac{m_G^2}{\Lambda_G^2} G^4 = T^\mu_\mu$$

The emergence of Λ_{YM} and of a gluon condensate is one of the most important things in the QCD vacuum... it has to do with gluons only. The scale in QCD is actually given by gluons!

The question is: do non-perturbative solutions such as instantons play a role for this?

$\Downarrow$

study of the instanton vacuum in QT.

Remember: instantons are $\overset{\text{finite action}}{\vee}$ classical solution of the Euclidean eom which describe the tunnelling probability between different vacua.

In particular, can instantons help to understand why a gluon condensate emerge?

Gluon condensate $\to$ even "before" the $\bar{q}q$ condensate...