't Hooft-Polyakov Monopole

The Dirac construction of a monopole is quite peculiar.

We had "a Dirac string" attached to the point-like monopole which is then "invisible" by shrinking the radius of the string to zero and by requiring that it is also "invisible" by the AB effect (which is a quantum property)

The Dirac Monopole is a "kind" of solitonic solution in which QM considerations entered. Moreover, its "mass" diverges classically, being a point-like object.

The question is: can we construct a magnetic monopole with finite energy and without the cumbersome introduction of Dirac strings?

The answer is indeed yes: the 't Hooft-Polyakov monopole of the Georgi-Glashow model.

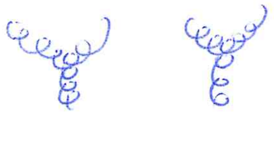
Recall: we are in our 1+3 dimensions.

Let us consider a SU(2) YM Lagrangian:

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\,\varepsilon^{abc} A^b_\mu A^c_\nu$$

These are "gluons" with self-interaction



The idea of 't Hooft and Polyakov is similar to the "vortex model" in 1+2 dimension that we studied some weeks ago, in which we had coupled a O(2) field to a U(1) e.m. field. Then, the idea was that a hedgehog configuration of the scalar field is "eaten up" by the e.m. field.

Here, we have a SU(2) field and we need to couple it to a O(3) scalar field.

in fact, A_μ^a $a = 1,2,3$ is the color index.

We thus introduce the Lagrangian (Georgi-Glashow)

$$\begin{cases} \mathcal{L} = \mathcal{L}_{YM} + \frac{1}{2}(D_\mu \phi^a)(D^\mu \phi^a) - \frac{\lambda}{4}\left(\phi^a\phi^a - F^2\right)^2 \\ D_\mu \phi^a = \partial_\mu \phi^a - ie\,\varepsilon^{abc} A_\mu^b \phi^c \end{cases}$$

$$\phi^a \mapsto B^{ab}\phi^b$$

$$A_\mu = A_\mu^a t^a \mapsto U A_\mu U^\dagger - \frac{i}{e} U \partial_\mu U^\dagger \qquad U = e^{i\alpha_a(x) t^a}$$

$t^a = \frac{\tau^a}{2}$ are the Pauli matrix

$$\phi = \phi^a t^a \mapsto U \phi^a U^\dagger$$

ϕ transform in the so-called adjoint representation.

Note. $\phi^a\phi^a = \mathrm{Tr}[\phi\phi] \mapsto \mathrm{Tr}[U\phi U^\dagger U \phi U^\dagger]$ ✓ invariant.

This theory shows, again, spontaneous breaking of a local gauge symmetry.

This is very similar as the process that we described one week ago in the "U(1)" case...

Still, the presence of SU(2) makes it a bit different.

One direction is "picked up", say $\phi^1 = F$, while $\phi^2 = \phi^3 = 0$. Then, one would have, without the gauge field, one massive "σ-field" and "two-massless" scalars.

But again, in the local gauge case we do not have Goldstone bosons but rather the generation of two massive gauge bosons...

We get here.

$$M_{W^\pm} = eF$$

Then, within this theory:

one massless gauge field $M_\gamma = 0$, two massive gauge bosons, $M_{W^\pm} = eF$, and one massive Higgs field, $M_H^2 = 2\lambda F^2$. $(\phi^1 = F + H)$

D.o.f counting:

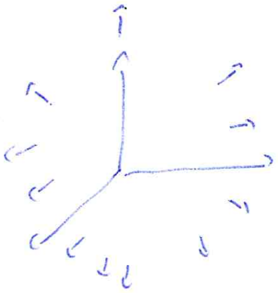
Perturbative: $\underset{3\gamma}{3\times2} + \underset{\phi^a}{3} = 9$

Non-perturbative: $\underset{\gamma}{2} + \underset{W^\pm}{3\times2} + \underset{H}{1} = 9$

This is even closer to the SM than what we have seen one week ago, although it is still different. Namely, in the SM the Higgs field is a doublet and is introduced in the fundamental (and not adjoint) representation.

Now, one writes down a static (i.e. solitonic) solution, which asymptotically behave as:

$$\phi^a \overset{r\to\infty}{\simeq} F\,\frac{x^a}{r}$$



This means, we go beyond the simple configuration $\phi^1=F,\ \phi^2=\phi^3=0$.

and

$$A_i^a \overset{r\to\infty}{=} -\varepsilon_{iab}\,\frac{x_b}{e r^2}$$

one can (relatively easily) show that $D_i\phi^a=0$, that is I can construct an object with finite energy

$$E=\int d^3x\,\Big[\underbrace{(D_i\phi^a)^2}_{\text{vanish}}+\underbrace{\tfrac{1}{2}F_{ij}^2}_{\text{vanish (pure gauge...)}}+\underbrace{V(\phi^a\phi^a)}_{\text{vanish}}\Big]$$

Still, this is a "good" configuration only for $r \gg R$, otherwise it is not, because it is singular for $r\mapsto 0$.

Anyhow, if we calculate the magnetic field we find

$$B_i^a = \frac{1}{e}\frac{x^a x^i}{r^4} \qquad \left(\sim \frac{1}{r^2} \text{ for large } r \dots\right)$$

which implies a magnetic charge equal to:

$$q_{mag} = g = \oint_{S_2} d\sigma_a B_a = \frac{4\pi}{e}$$

This means that we have also here a quantization rule

$$g e = 4\pi$$

(in general, $\boxed{g e = 4\pi n}$)

which is, apart from a factor 2, analogous to the one of Dirac. Besides that, the quantization arises naturally (no need of Dirac string here...).

Note, however, that we don't have only a magnetic monopole but also a Higgs field configuration attached to it.

But here one can go further; namely, one can obtain full solution (and not only asymptotic ones) of the static field configurations.

$$\phi^a \simeq F \frac{x^a}{F} \longmapsto \frac{x^a}{r} H(r)$$

$$A_i^a \simeq -\varepsilon_{iab} \frac{x_b}{e r^2} \longmapsto -\varepsilon_{iab} \frac{x_b}{e r^2} \left(1 - K(r)\right)$$

Determining these functions is not trivial; complicated eqs.

Solutions are only known for $\lambda \longmapsto 0$

$$\xi = Fer$$

$$\begin{cases} K \simeq \xi \operatorname{cosech} \xi \\ H = \xi \coth \xi - 1 \end{cases} \rightarrow$$ IMPORTANT: smooth everywhere!!! No singularity.

In this limit the mass of the field configuration:

$$M_{Mon} = \frac{4\pi F}{e}$$

e small $\longmapsto$ M_{Mon} is large.

For λ one can solve the problem numerically:

$$M_{Mon} \simeq \overset{\text{number larger than 1}}{\#} \left(\frac{4\pi F}{e}\right).$$

Note, one can reexpress the mass of the monopole as

$$M_{Mon} = \frac{4\pi}{e^2} M_{W^\pm}$$

↳ Masses of the massive gauge bosons...

That means also:

$$M_{Mon} = \frac{M_{W^\pm}}{\alpha}$$

if we are QED (in the U(1) "survival" symmetry, this theory be QED)

$$\alpha \simeq \frac{1}{137}$$

we then get

$$M_{Mon} \simeq 137\ M_{W^\pm}$$

Although this is not the SM, just for fun, setting $M_W \sim 90$ GeV, we get an astronomically high number.

→ Hard to condense... This is a problem if we want to implement the dual superconducting picture of confinement!

(condensation of magnetic charges ⟼ confinement of electric charges).
(Chromomagnetic) (Chromoelectric)

Concluding remark:

The 't Hooft - Polyakov monopole has not as energy...
its mass is finite.

The magnetic lines look like a monopole for large r,
but the field configuration is smooth.

It has a dimension, a finite size (and also a maximal
value of the magnetic field) just as it happens with the
gravitational field of the sun when its finite dimension R is
taken into account.

The finite size is, as expected, given by $\sim \frac{1}{M_W}$.

"Dyon" on PS magnetic monopole
(Prasad - Sommerfield)

$$\mathcal{L} = \mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$

We consider only the YM part.

The question is: can we construct a monopole in this theory?

Previously, in the Georgi-Glashow model there was also a scalar field ϕ^a attached to it...
Now there is not, so the 1° answer may be "no"... without ϕ^a we cannot have the cancellation of the two terms.

Still, previously we only had $A^a_i \neq 0$. A^a_0 was set to zero

(That is, no electric charge was considered).

Now, the idea is to consider $A^a_0 \neq 0$...

intuitively, A^a_0 takes the role of ϕ^a... in turn, this means that we have both nonzero (chromo) electric and magnetic fields.

$E^a_i \neq 0, \quad B^a_i \neq 0$

We will restrict to static configuration (which depend on $\vec{x}$ only). ?

Then, we make the following Ansatz: $\longrightarrow \mu$ = dimensionfull parameter

$$\begin{cases} A_0^a(\vec{x}) = (\mu r \coth \mu r - 1)\dfrac{x^a}{r^2} \\ A_i^a(\vec{x}) = (\mu r \,\mathrm{csch}\,\mu r - 1)\,\varepsilon_{iab}\dfrac{x^b}{r^2} \end{cases}$$

This solution has energy

$$M = 8\pi^2 \cdot \left(\frac{\mu}{2\pi}\right)$$

Asymptotically:

$$A_0^a \sim \left(\mu - \frac{1}{r}\right)\frac{x^a}{r}, \quad A_i^a \sim \varepsilon_{aij}\frac{x^j}{r^2}$$

$$\Downarrow$$

$$\begin{cases} E_i^a \sim \dfrac{x^a x^i}{r^4} & \left(\sim \dfrac{1}{r^2}\right) \\ B_i^a \sim -\dfrac{x^a x^i}{r^4} & \left(\sim \dfrac{1}{r^2}\right) \end{cases}$$

This is a Dyon; it has both the properties of an electric and a magnetic charge.

But there is a subtle point; in the GG model we were obliged to introduce a dimensionful quantity, which was the parameter $\underline{\underline{F}}$ (the 'circle' of the minima of the Mexican hat potential $V(|\phi^2|)$.

Now, there is nothing like F in a pure YM theory. Indeed, the only parameter is the dimensionless coupling constant e ( ).

But no dimension is in it...

If we calculate the mass of this configuration we get

$$M_{dyon} = 4\pi \frac{\mu}{e^2} \quad , \mu > 0 .$$

μ is a dimensionful parameter which we have introduced in the Ansatz of the solution.

It is necessary because the argument must be dimensionless, that is we have to multiply r by a dimensionful quantity, $r \to r\mu$.

The point is: μ is not determined. It is a 'open' quantity.

It can be whatever, it is not fixed... then, also M_{dyon} is not fixed.

This is indeed not a surprise, because $\mathcal{L}_{YM}$ is dilatation invariant.

(No dimension in it...)

Classically, there is no way how you can get a dimension in it.

You have to consider the quantum version of the theory for that purpose. So, at this classical level the study of dyon and monopole cannot be conclusive. We need to get a dimension in it...

Still, this field configuration shows that.

- we can construct a magnetic monopole in a YM theory

Although a quantum approach is necessary, this is an interesting property in view of the dual picture of confinement.

This then leads us to discuss:

- dimensional transmutation
- instantons

in YM theories.