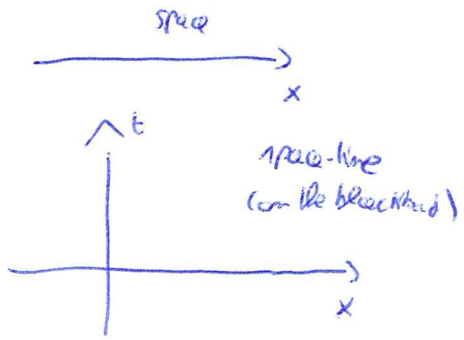
Starting point: lower dimensional field theory.

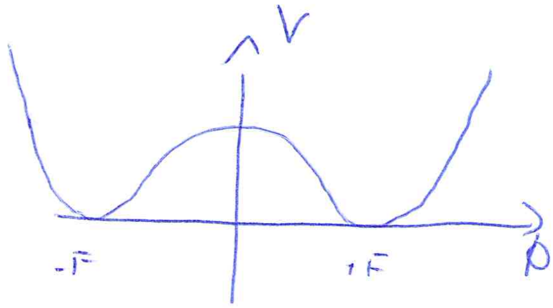


$$\underset{\text{time}}{\overset{\downarrow}{1}} + \underset{\text{space}}{\overset{\downarrow}{1}}$$

Many features can be seen in this simple context.

$\phi = \phi(t,x)$ $\rightarrow$ scalar field

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - F^2\right)^2$$



↳ Not only one vacuum, but two!!!

This is a general feature of all the problems that we will study.

Now, in 1+1 dimensions there are solutions with:

$$\begin{cases} \phi(t, x \mapsto -\infty) = -F \\ \phi(t, x \mapsto +\infty) = +F \end{cases}$$

"kink"

the simplest example of a soliton
"it carries a huge energy → a kind of "tsunami wave"".

- Introduction of the following concepts:

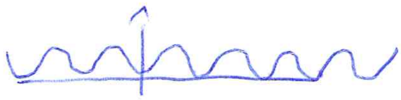
⇝ Topological charge (and current) → conserved quantity which arises

⇝ Bogomol'nyi inequality

⇝ (many kinks and antikinks

c

I will then move further to:

- 

infinitely many vacua ... even closer to "reality"

- A general theorem (Derrick-Hobart theorem) on the existence of solitons: if $D \geq 3$ no solitons exist if scalar fields only are used.

  → Is everything lost? → No, you need a richer structure (YM theory)

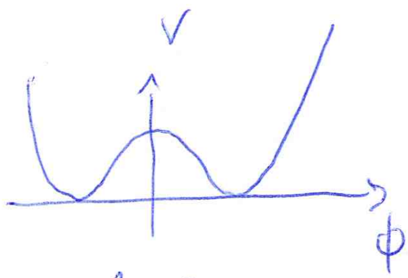
  → But before going to higher dimensions, I go to even lower one...

Next step: instanton

"instanton" = "localized finite-action solutions of the classical Euclidean field equations of a theory"

$D = 0$

$1+0 \rightarrow$ QM



$\phi = \phi(t)$

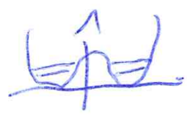
$\boxed{\tau = it}$

$\boxed{t^2 \mapsto -\tau^2.}$

→ QM is a field theory in 1+0 dimensions.

→ This theory admits "instantons", solutions → interpolation of the two vacua

→ relevant for the thermodynamics (... energy levels...)



S, A configuration

Tunnell

→ General case: soliton in D spatial dimensions → instantons in the very same theory in D − 1 spatial dimensions

(solution of the classical field theory)

(solution of the euclidean e.o.m, that is relevant for the QFT)!

Back to solitons and $D = 2$

$O(3)$ model in $1+2$ dimensions

$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ $\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi}^2$

but ...

$\boxed{|\vec{\phi}| = F}$ $\rightarrow$ constant length!!! Non trivial constraint!

This non trivial constraint changes the topology: the internal space

$O(3) \rightarrow$ in $1+1$ dimensions: Quantum.

Amazing similarities with YM-theory:

⤳ Trace anomaly

⤳ renormalizable

⤳ negative β function

⤳ Non-perturbative mass gap (Just a few plans...)

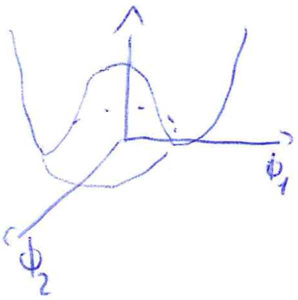
⤳ ... and instantons!!!

From solitons to monopoles

- Monopole in electrodynamics in $\underline{D=2}$ (and also in $D=3$). $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2}$
- $D=2$

1+2 and complex scalar field

$$V = \frac{\lambda}{4}\left(\phi^*\phi - F^2\right)^2$$



⇝ soliton solution which looks like a "vortex" (again, the presence of many vacua is crucial for all this)

⇝ It will be shown that this soliton solution in this model has indeed an infinite energy...

→ Introducing a further vector field one can obtain a stable vortex configuration and a stable

$$\mathcal{L} = [D_\mu\phi]^*[D^\mu\phi] - \frac{\lambda}{4}\left(\phi^*\phi - F^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Finally $\boxed{D=3}$ SO(1,3) $SU(2)$
proof

$$\mathcal{L} = -\frac{1}{4} F^{a\,2}_{\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{1}{4}\left(\vec{\phi}^2 - F^2\right)^2$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

↳ why "3"? Topology, matching of internal and external space

"t'Hooft-Polyakov Monopole"

$\boxed{D=3}$

SU(2) Yang-Mills

$$\mathcal{L} = -\frac{1}{4} G^{a\,2}_{\mu\nu}$$

→ Monopoles and also Instantons

From SU(2) to SU(3)

Quantisation of topolog. solutions

other topological objects (I will propose some, according to time and what you would like we will discuss some of them)

- Dyons (role of matter for theory)
- sphaleron (in SM)
- The proton as a topological object "made out of pions" (Skyrmion)
- Polyakov loop

...

...