Group $U(N)$:

$$U^\dagger U = U U^\dagger = 1 \qquad U = e^{-i\alpha^a t^a}$$

$a = 0, 1, \dots, N^2 - 1$

By convention:

$$t^0 = \frac{1}{\sqrt{2N}} 1_N$$

$$\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$$

$$[t^a, t^b] = i f^{abc} t^c$$

Group $SU(N)$

$$\begin{cases} U U^\dagger = U^\dagger U = 1 \\ \det U = 1 \end{cases}$$

$$U = e^{-i\alpha^a t^a} \text{ with } a = 1, \dots, N^2 - 1$$

Namely, out of $UU^\dagger = 1$ we get

$$\det(UU^\dagger) = \det U \det U^\dagger = \det U \det U^* = |\det U| = 1$$

which means $\det U = e^{i\varphi} \neq 1$ in general....

Example:

$$U = e^{-i\alpha^0 t^0}$$

$$\det U = e^{-i\alpha^0 \mathrm{Tr}[t^0]} = e^{-i\alpha^0 N/2\sqrt{N}} = e^{-i\alpha^0 \sqrt{N}/2} \neq 1!$$

Now, let us consider $SU(N)$ and concentrate our attention to a subgroup of it.

Let us study only a phase:

$$Z = e^{i\alpha} 1_N \left(= e^{i\alpha 1_N} = e^{i\alpha \sqrt{2N}\, t_0} \right)$$

$$Z^\dagger Z = Z Z^\dagger = (e^{-i\alpha} 1_N)(e^{i\alpha} 1_N) = 1_N \quad \checkmark$$

$$\det Z = (e^{i\alpha})^N = e^{iN\alpha} \quad (\neq 1 \text{ in general!})$$

If, now, α is "special" we can still get that $\det Z = 1$.

Namely, the following requirement must be fulfilled:

If $\alpha = \frac{2\pi n}{N}$ with $n = 0, 1, 2, \ldots, N-1$ $\rightarrow e^{iN\alpha} = e^{iN\left(\frac{2\pi n}{N}\right)} = e^{i2\pi n} = 1$

Note: we have to stop at $N-1$ in order to have N distinct elements...

In fact, $e^{iN\left(\frac{2\pi n}{N}\right)} = 1 \quad \forall n = 0, \pm 1, \pm 2, \ldots$ but they are not distinct.

For instance, for $n = 0$ and $n = N$ you get the same element; see below.

N elements:

$$z_m = e^{i\frac{2\pi m}{N}} 1_N$$

$$\underset{SU(N)}{C} = \{ \underset{m=0}{1_N}, \underset{m=1}{e^{i2\pi/N} 1_N}, \underset{m=2}{e^{i4\pi/N} 1_N}, \dots, \underset{m=N-1}{e^{i\frac{2\pi(N-1)}{N}} 1_N} \} = Z_N$$

[We see that if you put, say, $m = N$, you get $z_N = e^{i2\pi\frac{N}{N}} 1_N = 1_N$, which is just the first element]

C is a group:

There is an internal operation, there is the unity and the inverse....

Example:

$$z_1 \cdot z_2 = e^{i\frac{2\pi}{N}} 1_N \, e^{i4\pi/N} 1_N = e^{i\frac{2\pi}{N}(1+2)} 1_N = e^{i\frac{2\pi}{N}\cdot 3} 1_N = z_3$$

$z_1 \cdot$

which is the inverse?

$$z_1^{-1} = e^{-i2\pi/N} 1_N = e^{-i2\pi/N} \cdot e^{i\frac{2\pi}{N}N} 1_N = e^{i\frac{2\pi}{N}(N-1)} 1_N = z_{N-1}$$

To very beautiful mathematical properties.... still, the interesting fact is that they are relevant for Physics!!! It has to do with confinement!!!

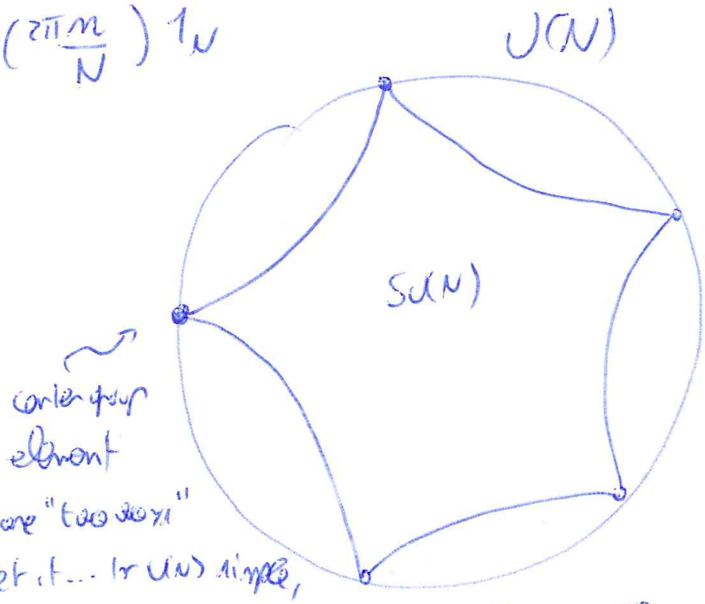
- C_N is a sub-group of $SU(N)$ ... and also of $U(N)$.

- In the framework of $U(N)$ is very simple to get a center member... just set $\alpha_1 = \alpha_2 = \dots \alpha_{N^2-1} = 0$ and set $\alpha_0 = \frac{2\pi n}{N} \cdot \sqrt{2N}$

$$U = e^{-i\alpha^a t^a} = e^{-i\alpha^0 t^0} = e^{+i(2\pi n/N) 1_N} = 1_N$$

However, this is also possible for a choice of $\alpha_1 = \alpha_2 \dots \alpha_{N^2}$ namely there is a special choice for which in $SU(N)$ only ($\alpha_0 = 0$)

$$e^{-i\alpha_a t^a} = e^{i\left(\frac{2\pi n}{N}\right) 1_N}$$



There are "two ways" to get it... In U(N) simple, in SU(N) more complicated, but obviously possible...

Explicit example: $N=2$

$$C_{SU(2)} = \{ 1_N , e^{i\pi} 1_N \} = \{ 1_N , -1_N \} = \mathbb{Z}_2$$

This is actually the smallest non-trivial group!

$$1_N = e^{-i\vartheta^a t^a} \text{ for } \vartheta^a = 0 !$$

In this case, by noticing that $t^a = \frac{\gamma^a}{2}$ where γ^a are the Pauli matrices, we have:

$$e^{-i\vartheta^a (\frac{\gamma^a}{2})} = e^{-i \frac{\vartheta}{2} \vec{n} \cdot \vec{\gamma}} \quad \text{where } \vec{n} \text{ is such that } |\vec{n}| = 1$$

$$= \gamma_0 \cos\frac{\vartheta}{2} - i (\vec{n} \cdot \vec{\gamma}) \sin\frac{\vartheta}{2}$$

Ergo, we see that for $\vartheta = 0$ we get $\boxed{1_2 = \gamma_0}$

but for $\vartheta = 2\pi$ we have:

$$e^{-i\vartheta^a t^a/2} = -1_2 \,!!!$$ (this is the other element of it...)

↳ Non trivial...

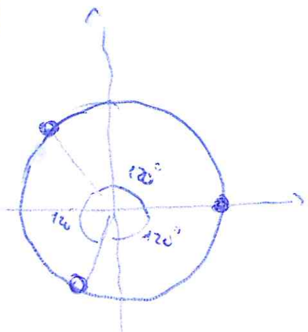
Note, this is independent on $\vec{n}$... so, if we take for instance $\vec{n} = (0,0,1)$ we have

$$e^{-i\frac{\theta}{2}Y_z} = Y_0 \cos\frac{\theta}{2} - i\, Y_z \sin\frac{\theta}{2} = -Y_0 \;!!! \quad \text{for } \theta = 2\pi.$$

[There is then an infinite possibilities to get -1_2.]

Explicit exemple: $N = 3$

$$\mathbb{Z}_3 \equiv C_{SU(3)} \equiv \left\{ 1_3 \, , \, e^{i 2\pi/3} 1_3 \, , \, e^{i 4\pi/3} 1_3 \right\}$$



$$e^{i 2\pi/3} = \cos\left(\frac{2}{3}\pi\right) + i \sin\left(\frac{2}{3}\pi\right) = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$$\frac{2}{3}\pi = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi + \pi}{6} = \frac{2}{3}\pi$$

$$e^{-i 2\pi/3} = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

Anyway, similar discussion as in the $N = 2$ case.

QCD Lagrangian for arbitrary N

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \qquad A_\mu = A_\mu^a t^a$$

$$A_\mu \mapsto U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$$

$$\begin{cases} \mathcal{L}_{QCD} = \mathcal{L}_{YM} + \bar{q}(\gamma^\mu D_\mu - m)q \qquad q \mapsto Uq \\ D_\mu = \partial_\mu - ie A_\mu \end{cases}$$

$\mathcal{L}_{QCD}$ is invariant under local gauge transf., $U = U(x) = e^{-i\theta^a(x) t^a}$

obviously, $\mathcal{L}_{QCD}$ is invariant also if $\qquad G_{\mu\nu} \mapsto U(x) G_{\mu\nu} U(x)^\dagger$

$U(x) = U = Z_n$!

This is trivial because : $\begin{cases} A_\mu \mapsto A_\mu \\ q \mapsto e^{i2\pi m/N} q \quad \rightarrow \bar{q}q \text{ is invariant.} \end{cases}$

Then, why all the discussion of center symmetry in QCD ?

In the vacuum everything is trivial.... but when going at nonzero temperature things change.

$$Z(T) = \int_{PBC} DA_\mu \, D\psi \, D\bar{\psi} \; e^{-S^E_{QCD}} \qquad (t = -i\tau)$$

$$S^E_{QCD} = \int_0^\beta d\tau \int d^3x \, \mathcal{L}^E_{QCD} \qquad \beta = \frac{1}{T}$$

PBC means:

$A_\mu(0,\vec{x}) = A_\mu(\beta,\vec{x})$ periodic for bosons

$\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$ antiperiodic for fermions (quarks)

Now, let us consider a τ dependent gauge transformation such that:

$U(0,\vec{x}) = 1_N$

$U(\beta,\vec{x}) = Z_m \mapsto$ center element with $Z_m \neq 1$.

$$A'_\mu(\tau,\vec{x}) = U A_\mu U^\dagger - \frac{i}{g} U \partial_\mu U^\dagger$$ suppose that the derivative vanish for $\tau = 0$ and $\tau = \beta$

$$\begin{cases} A'_\mu(0,\vec{x}) = U(0,\vec{x}) A_\mu(0,\vec{x}) U^\dagger(0,\vec{x}) = A_\mu(0,\vec{x}) \\ A'_\mu(\beta,\vec{x}) = Z_m A_\mu(\beta,\vec{x}) Z_m^\dagger = A_\mu(\beta,\vec{x}) \end{cases}$$

No problem... I can do this transformation, the YM part of the Lagrangian is fulfilled!!!

But for the quarks the situation is ≠!

$$\psi'(\tau,\vec{x}) = U(\tau,\vec{x})\,\psi(\tau,\vec{x})$$

$$\begin{cases} \psi'(0,\vec{x}) = \psi(0,\vec{x}) \\ \psi'(\beta,\vec{x}) = z_n\,\psi(\beta,\vec{x}) = -z_n\,\psi(0,\vec{x}) \end{cases}$$

it is <u>not</u> anymore symmetric...

Then we see, such a center transformation leaves $Z(T)$ invariant only in the YM sector but is not an invariant of full QCD.

Note, finite T is crucial for this... at zero T you don't have this issue...

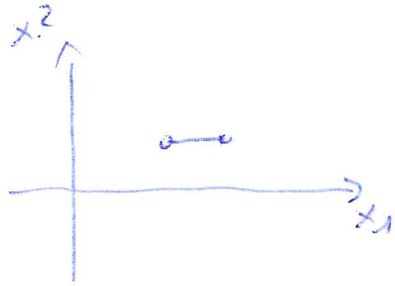
So, the center symmetry combined with local g.t. at nonzero T is the basis of the discussion. It is not center transformation by itself... one needs also the modified geometry of $T\neq0$ (thermal) and the (A)PBC of the fields.

Alternative way to the YM Lagrangian

$$\Omega_\mu(x) = e^{igA_\mu(x)dx^\mu}$$ Achtung: no sum over μ...

$$\Omega_1 = e^{igA_1 dx^1}$$

$\Omega_\mu(x)$ connect x^μ to $x^\mu + dx^\mu$ "gauge connection"



Now, we can write:

$\Omega_\mu = 1 + igA_\mu(x)dx^\mu$. because dx^μ is infinitesimal.

$\Omega_\mu \in SU(N)$ because $A_\mu = A_\mu^a t^a$

Ω_μ transforms as: $\Omega_\mu(x) \mapsto U(x)\,\Omega_\mu\, U^\dagger(x)$ (this is an hyp.)

[i.e. it transforms exactly as $q(x)\bar{q}(x+dx^\mu) \mapsto U(x)\, q(x)\bar{q}(x+dx^\mu)\, U$

[

Which is the transf. of A_μ in order this to be true?

$$\Omega_\mu = e^{iqA_\mu dx^\mu} = 1 + iqA_\mu dx^\mu \mapsto \Omega'_\mu = U(x)\left(1 + iqA_\mu dx^\mu\right)U^\dagger(x+dx^\mu)$$

$$\Omega'_\mu = 1 + iqA'_\mu dx^\mu$$

$$U(x+dx^\mu) = U(x) + \partial_\mu U \cdot dx^\mu$$

$$U^\dagger(x+dx^\mu) = U^\dagger(x) + \partial_\mu U^\dagger dx^\mu$$

Ergo:

$$U(x)\left(1+iqA_\mu dx^\mu\right)U^\dagger(x+dx^\mu) = U(x)U^\dagger(x+dx^\mu) + iq\big(U(x)A_\mu \underbrace{U^\dagger(x+dx^\mu)}_{U^\dagger(x)}\big)dx^\mu$$

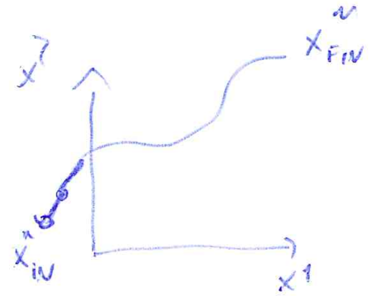
$$= 1 + U\partial_\mu U^\dagger \cdot dx^\mu + iqUA_\mu U^\dagger dx^\mu = 1 + iqA'_\mu dx^\mu$$

$$iqA'_\mu = iqUA_\mu U^\dagger + U\partial_\mu U^\dagger$$

$$\boxed{A'_\mu = UA_\mu U^\dagger - \frac{i}{q}U\partial_\mu U^\dagger}$$

one can also define the following Wilson line:

$$\mathbb{W} = P\, e^{\,i q \int_L A_\mu dx^\mu}$$



↳ Path ordering

$$= e^{iq A_\mu(x_{IN})\,\delta x^\mu_{IN}}\; e^{iq A_\mu(x_{IN}+\delta x^\mu_{IN})\cdot \delta x_2^\mu} \cdots\cdots e^{iq A_\mu(x_{FIN}-\delta x^\mu_{FIN})\,\delta x_{FIN}^\mu}$$

· How does this object transform under gauge transf?

$$\to U(x^\mu_{IN})\, e^{iq A_\mu(x_{IN})\,\delta x^\mu_{IN}}\, U^\dagger(x_{IN}+\delta x^\mu_{IN})\cdot U(x_{IN}+\delta x^\mu_{IN})\, e^{\cdots} \cdots$$

$$\cdot\, e^{iq A_\mu(x_{FIN}-\delta x^\mu_{FIN})\,\delta x^\mu_{FIN}}\, U^\dagger(x^\mu_{FIN})$$

Ergo

$$\mathbb{W} \mapsto U(x^\mu_{IN})\, \mathbb{W}\, U^\dagger(x^\mu_{FIN})$$

For $\mu = 0$ we have:

$$L(\vec{x}) = P e^{iq\int_0^\beta d\Upsilon A_4(\Upsilon, \vec{x})}$$

Polyakov line
(thermal Wilson line)

I simply integrate along the "0 = "4" direction.

Under a gauge transformation we have:

$$L(\vec{x}) \mapsto U(0,\vec{x}) L(\vec{x}) U^\dagger(\beta, \vec{x}) = Z_m^\dagger L(\vec{x}) = \overbrace{e^{-i2\pi m/N}}^{\text{phase}} L(\vec{x})$$

From:

$$P e^{iq\int_0^T dt A_0} \to P e^{-iq\int_0^\beta d\Upsilon A_4}$$

$$\begin{cases} t = -i\Upsilon \\ A^0 = -iA^4 \end{cases}$$

$$\boxed{dt A_0 = -d\Upsilon A_4}$$

Polyakov loop $\rightarrow$ trace over the Polyakov line

$$\ell = \frac{1}{N} \mathrm{Tr}\, L$$

Now, which is the expectation of the Polyakov loop? $\langle \ell \rangle$?

For very high T, $g \mapsto 0$, one expects $\langle \ell \rangle = 1$!

However, every other value of the center is allowed...

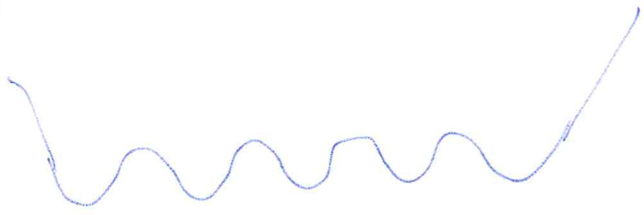
$$\langle \ell \rangle = e^{2\pi i n/N}$$

is possible for $T \mapsto \infty$.

One has a degeneracy.... all of them are ok.

The realized one is the usual phen. of spont. symm. breaking, low. realized at high T...

For high T $\mapsto$



N distinct minima!

Indeed this is also understandable in the following way: $\boxed{\langle l \rangle = e^{2\pi i m/N} l_0}$

l_0 real (for convention..)

$\dot{R}$ $\bar{R}$

$|\langle l_0 \rangle| \sim e^{-F_{rest}/T}$ $\to$ Free energy...

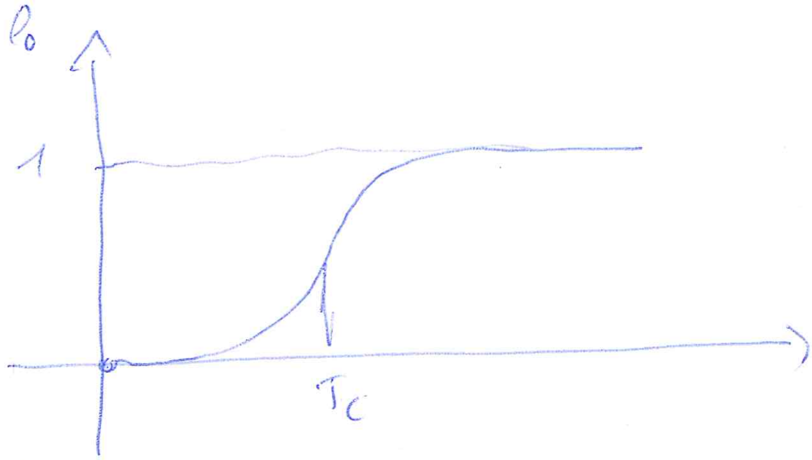
$|\langle l \rangle|$ is finite at high T $\mapsto$ F_{rest} is finite

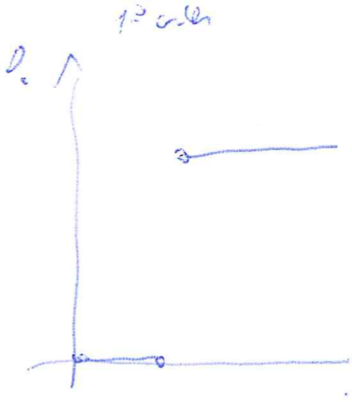
In the case
$|\langle l_0 \rangle| = 1$ $\to F_{rest} = 0 \to$ deconfinement!

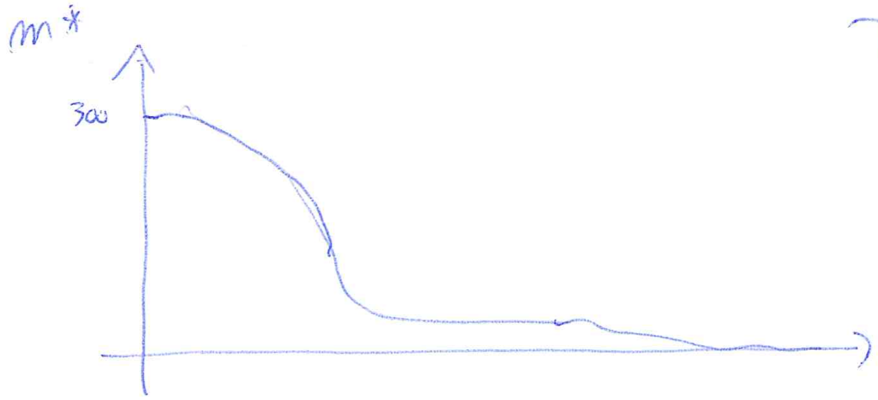
Then, for small T we observe confinement:

$|\langle l_0 \rangle| = 0$ because $F_{rest} = \infty$!!!

Then we expect that $\langle l_0 \rangle$ is 0 at $T \neq 0$, up at high T



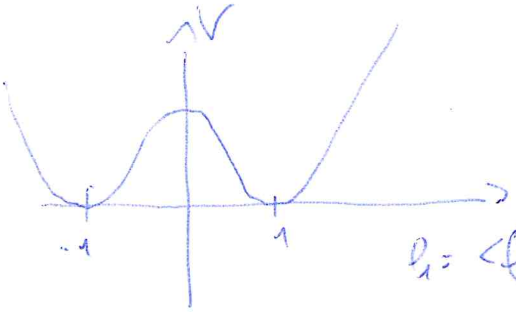




$\to$ it depends on N, on the presence of quarks

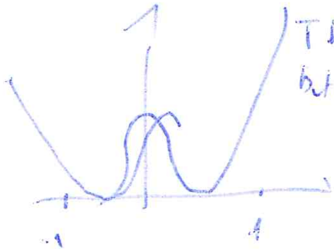
For $N=2$ the center is $\{1_2, -1_2\}$

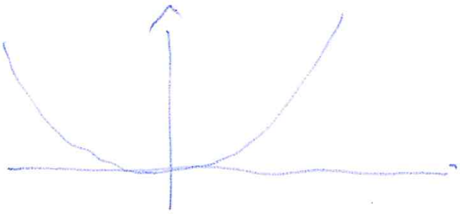
$\langle \ell \rangle$ at high T is either 1 or -1.



$T \mapsto \infty$

$\ell_1 = \langle \ell \rangle = 1$ or -1





$T \mapsto 0$

$\langle \ell \rangle = 0$

$$\boxed{V = \frac{m_1^2}{2}\ell_1^2 + \frac{\lambda}{4}\ell_1^4}$$

Simmetric under $\ell_1 \mapsto -\ell_1$, which is in $Z(2)$

$N=3$

$\langle \ell \rangle = \ell_0$

$\langle \ell \rangle = e^{i2\pi/3}\,\ell_0$ $\rightarrow$ These are the minima of the potential...

$\langle \ell \rangle = e^{i4\pi/3}\,\ell_0$

ℓ_1 is now complex

$$V_1 = m_1^2 |\ell_1|^2 + \kappa_1\left(\ell_1^3 + (\ell_1^*)^3\right) + \kappa_1\left(|\ell_1|^2\right)^2$$

Note, if I send:

$\ell_1 \mapsto e^{i2\pi/3}\,\ell_1$ $\rightarrow$ the first and the third terms are invariant!

What about the second?

$$\ell_1^3 \mapsto \left(e^{i2\pi/3}\,\ell_1\right)^3 = \left(e^{i2\pi}\,\ell_1\right)^3 = \ell_1^3 \qquad \text{OK}$$

$\ell_1^{*3} \mapsto$ the same

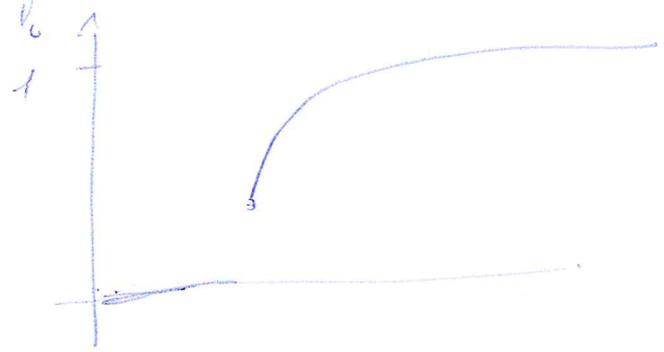
$\ell_1^3 + \ell_1^{*3} \mapsto$ ensure that it is real.

$\Rightarrow$ 1° order phase transition is realised here...

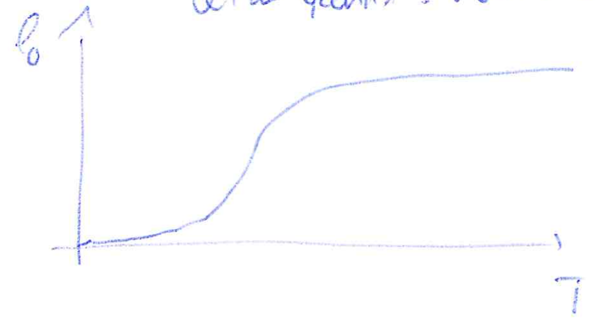
$N=3$ 1° order for YM

But only a cross-over if

without quarks : 1st order



with quarks : cross over



PNJL model : contains at the same time both "χSB and PL"