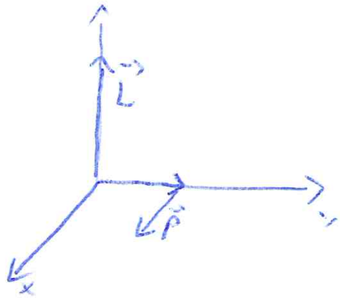
Angular momentum for one particle:



$$\vec{L} = \vec{r} \times \vec{p}$$

$$|\vec{L}| = r p \sin\theta_{\vec{r},\vec{p}} = r m v \sin\theta_{\vec{r},\vec{p}}$$

$\vec{L} \perp$ to the plane of motion

For a system of N particles

$$\vec{L}_{tot} = \sum_{i=1}^{N} \vec{L}_i$$

$$\boxed{\frac{d\vec{L}_{tot}}{dt} = 0}$$

For a single particle:

$$\frac{d\vec{L}}{dt} = 0 = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{\substack{=0 \\ \text{always}}} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = 0 \qquad \vec{F} \propto \vec{r}$$

$\vec{L}$ = const $\rightarrow$ The motion is on a plane.

Components:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} =$$

$$= \vec{i}\left(y p_z - z p_y\right) - \vec{j}\left(x p_z - z p_x\right) + \vec{k}\left(x p_y - y p_x\right)$$

$$\begin{cases} L_x = y p_z - z p_y \\ L_y = -x p_z + z p_x \\ L_z = x p_y - y p_x \end{cases}$$

or in a short-hand notation

$$\boxed{L_i = \varepsilon_{ijk} x_j p_k}$$

QM/1

$$\begin{cases} x, y, z \longmapsto \hat{x}, \hat{y}, \hat{z} \\ p_x, p_y, p_z \longmapsto \hat{p}_x, \hat{p}_y, \hat{p}_z \end{cases} \qquad \hat{p}_k = -i\hbar \frac{\partial}{\partial x_k} = -i\hbar \partial_k$$

Recall.

$$[\hat{x}, \hat{p}_x] = i\hbar \qquad \hat{p}_x = -i\hbar\partial_x$$

$$(\hat{x}\hat{p}_x - \hat{p}_x x) f(\vec{x}) = x(-i\hbar\partial_x f) - (-i\hbar\partial_x)(x f(\vec{x})) =$$

$$= \cancel{-i\hbar x \partial_x f} + i\hbar f + \cancel{i\hbar x \partial_x f} = i\hbar f(\vec{x})$$

Now:

$$[\hat{x}, \hat{p}_y] = 0$$

$$(x p_y - p_y x) f(\vec{x}) = x(-i\hbar\partial_y f) - (-i\hbar\partial_y)(x f(\vec{x})) = 0!$$

Summarizing:

$$\begin{cases} [\hat{x}_i, \hat{x}_j] = 0 \\ [\hat{p}_i, \hat{p}_j] = 0 \\ [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \end{cases}$$

QM/2

$$L_x \mapsto \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

is it Hermitian?

$$\hat{L}_x^{+} = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)^{+} = \hat{p}_z^{+}\hat{y}^{+} - \hat{p}_y^{+}\hat{z}^{+} = \hat{p}_z\hat{y} - \hat{p}_y\hat{z} \equiv$$

$$\equiv \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = L_x \ !$$

$\hat{L}_x$ is herm. This is so because I couple $\neq$ indices.

(remember: $\hat{x}\hat{p}_x$ is not... $\frac{1}{2}(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$ is hermitian).

of course:

$$\hat{L}_x^{+} = \hat{L}_x$$

$[L_x, L_y]$ (cont 1)

$L_x = yP_z - zP_y$

$L_y = -xP_z + zP_x$

$L_z = xP_y - yP_x$

$$[\hat{L}_x, L_y] = [yP_z - zP_y, -xP_z + zP_x] =$$

$$= [yP_z, -xP_z] + [yP_z, zP_x] + [-zP_y, -xP_z] + [-zP_y, zP_x].$$

1st term:

$$[yP_z, -xP_z] = y[P_z, -xP_z] + [y, -xP_z]P_z =$$

$$= -yx\underbrace{[P_z, P_z]}_{=0} + y\underbrace{[P_z, -x]}_{=0}P_z + (-x)\underbrace{[y, P_z]}_{=0}P_z + \underbrace{[y, -x]}_{=0}P_zP_z$$

$$= 0 \,!$$

2nd term:

$$[yP_z, zP_x] = y[P_z, z]P_x = -i\hbar\, yP_x$$

3rd term:

$$[-zP_y, -xP_z] = x[z, P_z]P_y = i\hbar\, xP_y$$

4th term:

$$[-zP_y, zP_x] = 0$$

Put it together:

$$[L_x, L_y] = i\hbar\,(x P_y - y P_x) = i\hbar L_z$$

Presence of i = obvious (commutator of two herm. operators is antihermitian).

$[L_x, L_y] \neq 0$ (no comm. ... I do not have a common basis).

Generalization:

$$\boxed{[L_i, L_j] = i\hbar\,\varepsilon_{ijk} L_k}$$

crucial relation of angular-momentum operators.

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Note, the indet. relation implies that:

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2}\,|\langle L_z\rangle|$$

Note, I can have $\Delta L_x \cdot \Delta L_y = 0$ if $\langle L_z\rangle = 0$.

Even the case $\Delta L_x = \Delta L_y = 0$ is indeed possible, but corresponds $\psi = \psi(r)$.

classically: $|\vec{L}|$ is the modulus ... $\vec{L}^2$ is the square of it.

for QM we have the operator

$\hat{\vec{L}}^2 \equiv \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

This is a well-defined operator in QM.

Now, we can show that $[L^2, L_z] = 0$.

$[L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + \underbrace{[L_z^2, L_z]}_{=0} =$

$= L_x [L_x, L_z] + [L_x, L_z] L_x + L_y [L_y, L_z] + [L_y, L_z] L_y$

$= L_x(-i\hbar L_y) + (-i\hbar L_y) L_x + L_y(+i\hbar L_x) + (i\hbar L_x) L_y =$

$= 0$!!!

$$\boxed{[L^2, L_i] = 0}$$

Fact: I can chose a common basis for L^2 and L_z (or any other...)

The choice (L^2, L_z) is the one which is usually done.

Possible ex: other commutation relations

$$[L_i, X_j] = i\hbar \varepsilon_{ijk} X_k$$

$$[L_i, P_j] = i\hbar \varepsilon_{ijk} P_k$$

it follows that:

$$[L_i, \vec{X}^2] = 0$$

$$[L_x, x^2+y^2+z^2] = [L_x, y^2] + [L_x, z^2] =$$

$$= y[L_x, y] + [L_x, y]y + z[L_x, z] + [L_x, z]z$$

$$= y(i\hbar z) + (i\hbar z)y + z(-i\hbar y) + (-i\hbar \cancel{x})z$$

$$= 0!$$

Similarly:

$$[L_i, \vec{P}^2] = 0 \, .$$

Important Consequence:

$$H = \frac{\vec{P}^2}{2m} + V(r) \qquad r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$$

Then:

$$[L_i, H] = \underbrace{\left[L_i, \frac{\vec{P}^2}{2m}\right]}_{=0} + \underbrace{[L_i, V(r)]}_{=0} = 0!!!$$

It means that we have that L_i and H can be associated to the same basis.

In these kinds of problems it is a convention to chose to diagonalize

$$H, L^2, L_z$$

$$\psi_{nlm}(x,y,z) \equiv \psi_{nlm}\underbrace{(r, \theta, \varphi)}_{\text{Polar coordinates.}}$$

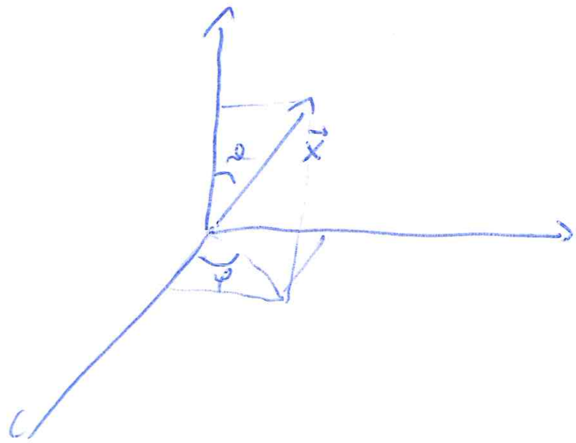
$$\begin{cases} H\psi_{nlm} = E_n \psi_{nlm} \\ L^2 \psi_{nlm} = \hbar^2 l(l+1)\psi_{nlm} \\ L_z \psi_{nlm} = \hbar m \psi_{nlm} \end{cases}$$

L_z : polar coordinates

In problems in which rotations are involved, the use of polar coordinates is typically very usefull.

In particular, L_z takes a simple form in polar coordinates.

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$



invert:

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \arctan \frac{y}{x} \\ \theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

How is the operator $\frac{\partial}{\partial x}$ in polar coordinates?

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \vartheta}{\partial x}\frac{\partial}{\partial \vartheta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

(a rather complicated expression indeed :))

$$\boxed{L_z = x P_y - y P_x \equiv -i\hbar \frac{\partial}{\partial \varphi}}$$

Verify (it is more simple ...)

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} + \underbrace{\frac{\partial z}{\partial \varphi}}_{=0}\frac{\partial}{\partial z}$$

$$= -r \sin\vartheta \sin\varphi\, \partial_x + r \sin\vartheta \cos\varphi\, \partial_y$$

$$= -y\,\partial_x + x\,\partial_y$$

$$-i\hbar\,\partial_\varphi = -y(-i\hbar\,\partial_x) + x(-i\hbar\,\partial_y) = x P_y - y P_x = L_z$$

qed (equivalent!!!!)

Eigenvalue eq. for L_z

$$\hat{L}_z \psi(\varphi) = \hbar m \psi(\varphi) \qquad p_z = \hbar m$$

$\psi(\varphi) = e^{im\varphi}$ is an eigenfunction.

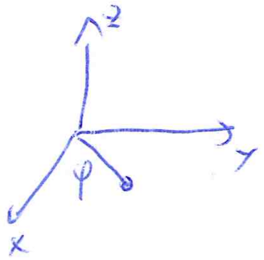
$$\left[\psi(r,\vartheta,\varphi) = e^{im\varphi} f(r,\vartheta)\right]$$

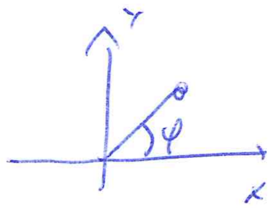
Namely:

$$L_z \psi(\varphi) = (-i\hbar)(im)e^{im\varphi} = \hbar m e^{im\varphi}.$$

There is a subtle issue: φ is an angular variable.

$$\psi(\varphi) = \psi(\varphi + 2\pi)$$





$$e^{im\varphi} = e^{im(\varphi+2\pi)}$$

$$1 = e^{im2\pi} \longrightarrow m = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$\boxed{m \text{ INTEGER}}$

The eigenvalues of L_z are $\hbar m$ with m being an integer.

Thus, I have multiples of QM.

All this is for the 'special' angular momentum correct.
However, in QM there is also an intrinsic angular momentum,
called spin.. for that it will be, that the state

$$|spin\rangle \xrightarrow{\text{rotation } 2\pi} \pm |spin\rangle \qquad (\text{Just as } \psi(\varphi))$$

$s = 0, 1, 2, \ldots$ bosons

$$|spin\rangle \xrightarrow{\text{rotation}} -|spin\rangle$$

$s = \frac{1}{2}, \frac{3}{2}, \ldots$ ↳ fermions

Naive:

$$e^{im\varphi} = -e^{+im(\varphi+2\pi)} \rightarrow e^{im2\pi} = -1$$

$$m = \frac{1}{2}, \frac{3}{2}, \ldots$$

↳ this alone is itself not a problem because only $|\psi|^2$ is a physical quantity!

This is only a digression about spin with heuristic arguments....

Question :

$[\varphi, L_z] = i\hbar$ $\qquad$ $L_z = -i\hbar\partial_\varphi$

one would get

$$\Delta\varphi\,\Delta L_z \geq \frac{\hbar}{2}$$

But $\Delta\varphi \leq 2\pi$ ... $\qquad$ if $\Delta L_z = 0$, $\Delta\varphi = 2\pi$ ...

$$2\pi \cdot 0 = 0 \geq \frac{\hbar}{2} \longrightarrow \hbar = 0 \text{ !!!}$$

Where is the error ????

Reason:

φ is not a well-defined operator:

$\psi(\varphi)$ / $\psi(\varphi) = \psi(\varphi + 2\pi)$

but:

$\varphi\,\psi(\varphi) = f(\varphi)$ is not such.

$f(\varphi) = \varphi\,\psi(\varphi)$

$f(\varphi + 2\pi) = (\varphi + 2\pi)\,\psi(\varphi + 2\pi) = (\varphi + 2\pi)\,\psi(\varphi) = \overbrace{\varphi\,\psi(\varphi)}^{f(\varphi)} + 2\pi\,\psi(\varphi)$

$\neq f(\varphi)$.

One can solve this problem by considering a $\neq$ operator than φ.

still, if I perform measurements I get some results for "φ" and some results for "L_z" ....

$\Delta\varphi \Delta L_z$ can be small as I want. $\Delta\varphi$ is "measurable !!!!"

for $\psi(\varphi) = e^{im\varphi}$ all the results for φ are equiprobable.

L^2

Operator $L^2 = \vec{L}^2$

$$\begin{cases} L^2 = -\hbar^2 \left\{ \frac{1}{\sin\theta} \partial_\theta \left[\sin\theta \, \partial_\theta \right] + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} = L_x^2 + L_y^2 + L_z^2 \\ L_z = -i\hbar \partial_\varphi \end{cases}$$

One finds a set of eigenstates of both L^2 and L_z with:

$$L^2 Y_{\ell m}(\theta,\varphi) = \hbar^2 \ell(\ell+1) Y_{\ell m}(\theta,\varphi) \Rightarrow |\vec{L}| = \hbar\sqrt{\ell(\ell+1)}$$ (Semiclassic)

$$L_z Y_{\ell m}(\theta,\varphi) = \hbar m Y_{\ell m}(\theta,\varphi) \qquad L_z = \hbar m$$

with $m = -\ell, -\ell+1, \dots -1, 0, 1, \dots, \ell$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\varphi}$$

$$Y_{1,-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

In general:

$$\boxed{Y_{\ell m} = N_{\ell m} \sin\theta^{m} \left[\left(\frac{d}{d\cos\theta} \right)^{\ell+m} (\sin\theta)^{2\ell} \right] e^{im\varphi}}$$

$$N_{\ell m} = \frac{(-1)^{\ell+m}}{2^{\ell}\ell!} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}$$

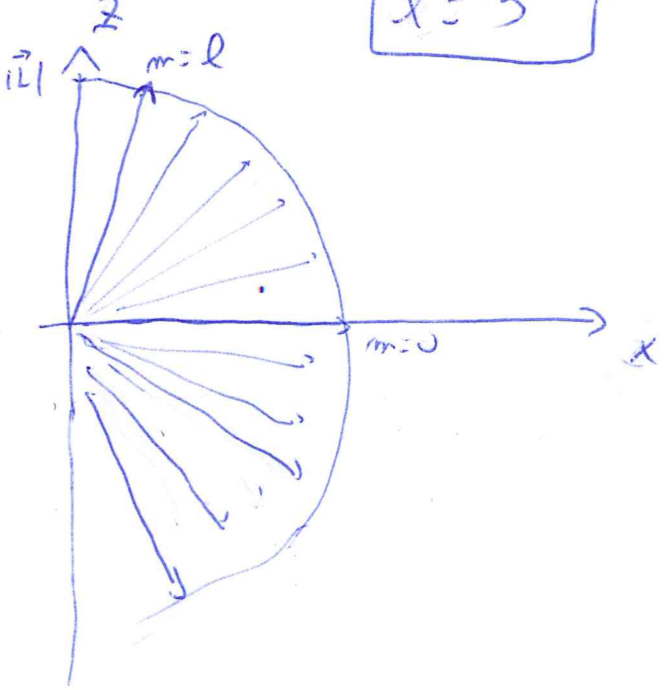
$$\int d\Omega \, Y^{*}_{\ell m} Y_{\ell' m'} = \delta_{\ell\ell'} \delta_{mm'}$$

Heuristic discussion:

For fixed l the full angular momentum (in modulus) is

$|\vec{L}| = \hbar\sqrt{l(l+1)} \simeq \hbar l$ (only for large l)

$\boxed{l = 5}$



$L_z^{MAX} = l \leq \hbar|\vec{L}|$

$\vec{L}$ in QM is not exactly on the z direction... If it were so I would have $L_x = L_y = 0$, but the latter do not commute!!