Indet. relation.

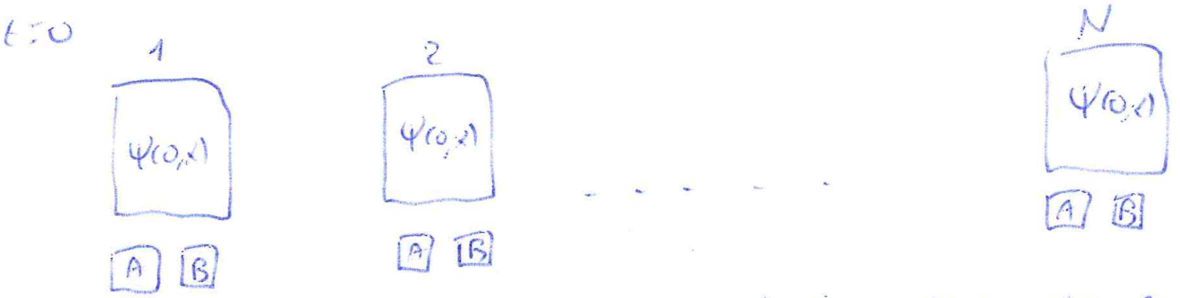
$\hat{A}, \hat{B}$ operators in $L^2(-\infty, \infty)$.

If $[\hat{A}, \hat{B}] = 0 \leftrightarrow \{\varphi_n\}$ such that
$$\hat{A}\varphi_n = a_n \varphi_n$$
$$\hat{B}\varphi_n = b_n \varphi_n$$

If $[\hat{A}, \hat{B}] \neq 0$ there is no such basis.

Let $\psi(0,x) = f(x)$ be the wf of our particle at time $t=0$.

We can prepare this state in many different boxes



Then, we have for each box two detectors. Yet, we make a meas. of either A or B (not both at the same time, that is not possible always one after the other. And after we have meas. A, we have a collapse and a fully new state. Moreover, for $[\hat{A}, \hat{B}] \neq 0$, the order is important: first $\hat{A}$ and then $\hat{B}$ is $\neq$ from first $\hat{B}$ and $\hat{A}$:)

Then, after we perform -say- in $N/2$ of cases a meas. of A and in the other $N/2$ a meas. of B, we get some results for $\langle \hat{A} \rangle_\psi$ and $\langle \hat{B} \rangle_\psi$.

The theory of QM predicts:

$$\langle \hat{A} \rangle_\psi = \int_{-\infty}^{\infty} dx\, \psi^*(0,x)\left(\hat{A}\psi(0,x)\right) = \left(\psi(0,x), \hat{A}\psi(0,x)\right)$$

$$\langle \hat{A}^2 \rangle_\psi = \int_{-\infty}^{\infty} dx\, \psi^*(0,x)\, \hat{A}\left(\hat{A}\psi(0,x)\right) = \left(\psi(0,x), \hat{A}^2\psi(0,x)\right)$$

$$\langle \hat{B} \rangle_\psi = \int_{-\infty}^{\infty} dx\, \psi^*(0,x)\left(\hat{B}\psi(0,x)\right) = \left(\psi(0,x), \hat{B}\psi(0,x)\right)$$

$$\langle \hat{B}^2 \rangle_\psi = \int_{-\infty}^{\infty} dx\, \psi^*(0,x)\left(\hat{B}^2\psi(0,x)\right) = \left(\psi(0,x), \hat{B}^2\psi(0,x)\right)$$

I can compare th. vs. exp. QM works (done many many times...).

QM does <u>not</u> predict the outcome of the single meas. with certainty (unless the state is an eigenstate), but the averages are well defined predictions of quantum theory.

standard deviations:

$$(\Delta \hat{A})_\psi = \sqrt{\langle \hat{A}^2 \rangle_\psi - \langle \hat{A} \rangle_\psi^2}$$

$$(\Delta \hat{B})_\psi = \sqrt{\langle \hat{B}^2 \rangle_\psi - \langle \hat{B} \rangle_\psi^2} \qquad \psi = \psi(0, x).$$

One can mathematically show that

$$\boxed{(\Delta \hat{A})_\psi (\Delta \hat{B})_\psi \geq \frac{1}{2} \left| \langle i [\hat{A}, \hat{B}] \rangle_\psi \right|}$$

indet. relation. Well fulfilled by exp. data.

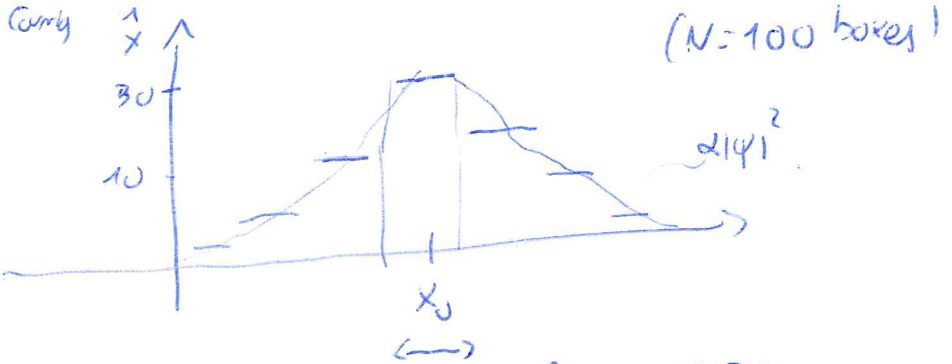
$\hat{A} = \hat{x}$, $\hat{B} = \hat{p}$ $\qquad [\hat{x}, \hat{p}] = i\hbar$ (constant)

$$\boxed{(\Delta \hat{x})_\psi (\Delta \hat{p})_\psi \geq \frac{\hbar}{2}}$$

Note, we never need to measure both $\hat{x}$ and $\hat{p}$ on the same particle to obtain (and verify) this relation.

Note

$(\Delta\hat{x})_\psi$ is the standard deviation of $\hat{x}$ ... it is not the "error" of the measurement. Actually these are expressions of theory, there is nothing like an exp. error.



δx is the exp. error

In order to see a nice peak: $\delta x << (\Delta\hat{x})_\psi$.

Alternatively, we could have meas. $\hat{p}$ with similar considerations.

$$\delta p << (\Delta\hat{p})_\psi .$$

$\delta p \cdot \delta x \Rightarrow$ NO RESTRICTION (CAN BE ALSO $< \hbar/2$).

Digression

Heisenberg microscop

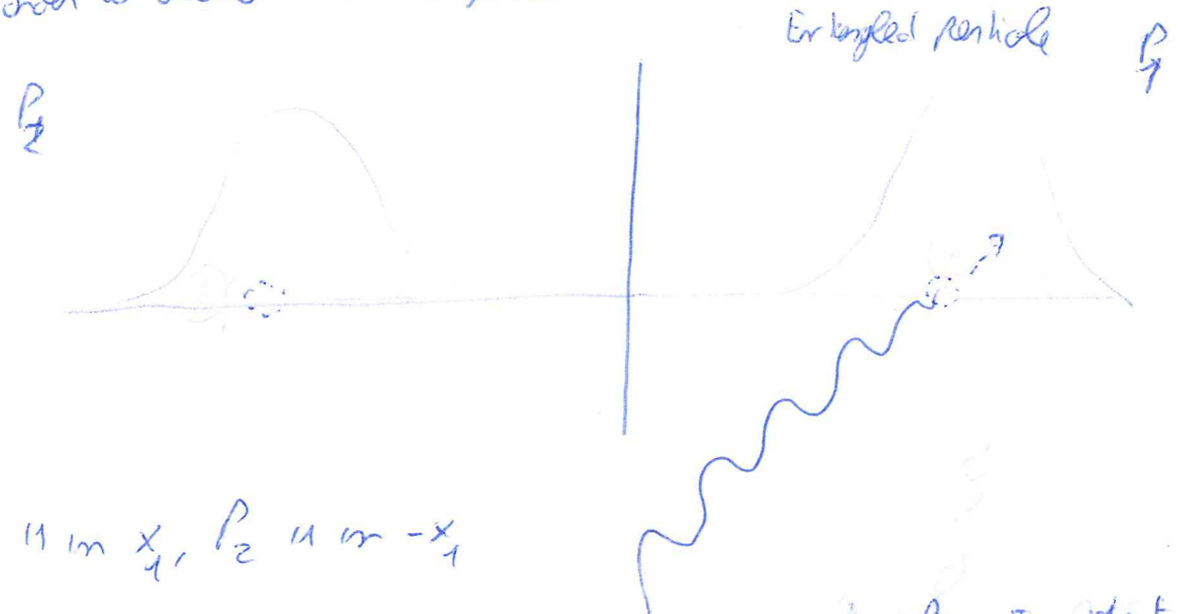
$\delta x \sim \lambda$

But by measuring I change p by $\Delta p \sim p_\gamma = \frac{h}{\lambda}$. The particle gets a "hit" and:

$\delta x \, \delta p \sim h$.

This is surely correct. But it is "completely" $\neq$ from the previous rigorous mathematical discussion.

In order to avoid this argument:



If P_1 is in x_1, P_2 is in $-x_1$

Now, if we measure P_1 $\mapsto$ we get $x_1 \pm \delta x_1$, then I disturb it ... but I also see that P_2 is in $-x_1 \pm \delta x_1$ without kicking it away!

The problem of Heisenberg is actually a technicality related to the peculiar way in which it is defined.

Apparently discussion is the following.

Suppose that

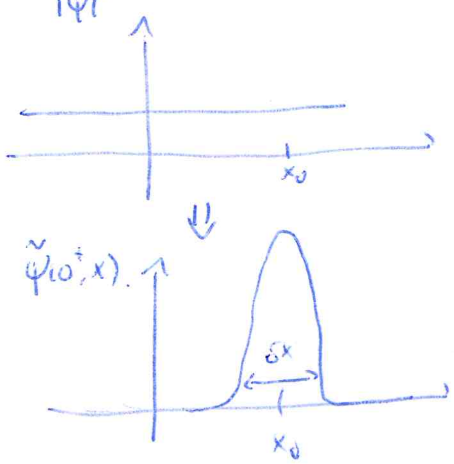
$$\psi(0,x) = e^{i p_i x}$$

We measure $\hat{p} \mapsto p_i$ is the result.

Then, if we measure $\hat{x}$ we get a certain x_0 with a error δx.

(Note = each x_0 in the box is equally probable.)

Then the collapse is:



$t = \epsilon$ measure $\hat{p}$

$\longrightarrow$

If at $t = \epsilon^+$ I meas. p (remind Gauss) I will find a certain p_f (with some δp_f).

The smaller δx, the more spread is p_f...

$$\underbrace{(p_f - p_i)}_{\text{this is novelty} \neq \text{indeed...}} \sim \frac{\hbar}{\delta x}$$

$\hookrightarrow$ this is indeed the error

Time and continuity eq.

Up to now I did not discuss the time evolution (I mentioned it but I did not present it).

I always introduced $\psi(0,x)$ and discussed operat./measurements on it.

For generality: $x \mapsto \vec{x}$.

The Schrödinger eq. for a particle with mass m and subject to the potential $V(\vec{x})$ is given by:

(1) $$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi}$$

$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$

$V = V(\vec{x})$ (real)

$\psi(0,x) \longmapsto \psi(t,x)$ is known for each "t".

Fully deterministic (up to the mass).

$*$

(2) $$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi^* + V\psi^*$$

Then, we multiply (1) by ψ^* and (2) by ψ:

$$\begin{cases} i\hbar\, \psi^* \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m} \psi^* \Delta \psi + V \overbrace{\psi^* \psi}^{|\psi|^2} \\[2ex] -i\hbar\, \psi \dfrac{\partial \psi^*}{\partial t} = -\dfrac{\hbar^2}{2m} \psi \Delta \psi^* + V \underbrace{\psi \psi^*}_{|\psi|^2} \end{cases}$$

subtract:

$$i\hbar \left(\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \right) = -\frac{\hbar^2}{2m} \left(\psi^* \Delta \psi - \psi \Delta \psi^* \right)$$

$$i\hbar \frac{\partial (|\psi|^2)}{\partial t} = -\frac{\hbar^2}{2m} \underbrace{\vec{\nabla} \cdot \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)}_{\cancel{\vec{\nabla}\psi^* \cdot \vec{\nabla}\psi} + \psi^* \Delta \psi - \cancel{\vec{\nabla}\psi \cdot \vec{\nabla}\psi^*} - \psi \Delta \psi}$$

✓

Introduce: $\rho = |\psi|^2$

$$\vec{J} = \frac{i\hbar}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J} \qquad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

$$\begin{cases} \rho = \text{density} \\ \vec{J} = \text{flux} \end{cases}$$

... but of what ??? of probability !!!!!

$\rho(t,\vec{x}) = |\psi(t,\vec{x})|$ is the probability density of position.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad \int \frac{\partial \rho(t,\vec{x})}{\partial t} d\vec{x} = \boxed{\frac{d}{dt} \int d^3x \, \rho(t,\vec{x})}$$

$\int_V d^3x$

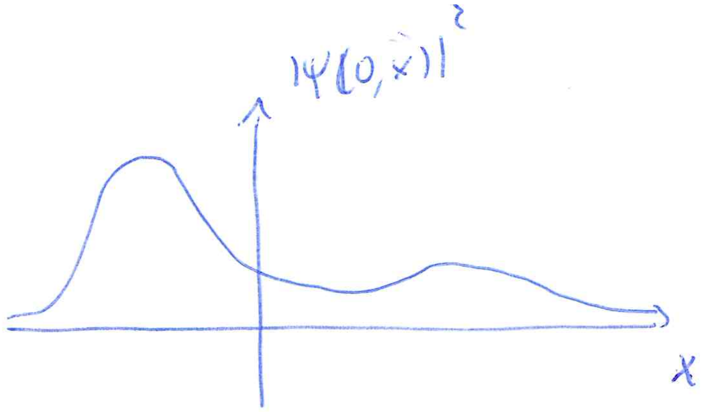
$$\frac{d}{dt} \int_V d^3x \, \rho = -\int_V \vec{\nabla} \cdot \vec{J} = -\int_{S_V} \vec{J} \cdot \vec{n} \, d\sigma$$

For $V \mapsto V = \infty \rightarrow = 0$

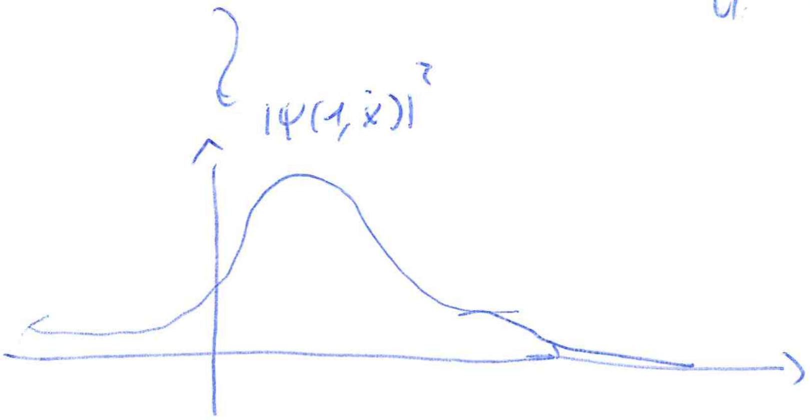
$$\underbrace{\frac{\partial}{\partial t} \int_U d^3x \, \rho = 0}$$

If $\int d^3x \, \rho(t,\vec{x}) = 1$ for $t = t_0 \Rightarrow$ it is such for each t.

This is a necessary requirement for a prob. interpretation.



$$\psi(0,x) \overset{\text{Schr.}}{\Downarrow} \psi(1,x)$$



The "whole wave has moved" and changed shape, <u>but</u> the area is still 1! Obvious: somewhere we must still have the particle.

Interesting alternative expression of the wf:

$$\psi(t,\vec{x}) = R(t,\vec{x})\, e^{i\phi(t,\vec{x})}$$

it follows that:

$$\rho(t,\vec{x}) = |R(t,\vec{x})|^2$$

$$\vec{J} = \frac{-i\hbar}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = \mathrm{Re}\left(\frac{\hbar}{im}\psi^*\vec{\nabla}\psi\right)$$

$$= \rho\,\vec{\nabla}\left(\frac{\hbar}{m}\phi\right) = \rho\frac{\hbar}{m}\vec{\nabla}\phi$$

($\vec{J} \propto$ grad of the phase)

[Note: we can also rewrite ψ as f of R and ϕ]

Note 2:

$(\vec{J}_e = \rho_e\vec{v})$

(De Broglie)

$\vec{v} = \frac{\hbar}{m}(\vec{\nabla}\phi) \mapsto$ Bohm mechanics

"Real trajectories"

In 3d

$$\begin{cases} \hat{\vec{P}} \equiv -i\hbar\vec{\nabla} = (\hat{P}_x, \hat{P}_y, \hat{P}_z) \\ \hat{\vec{X}} \equiv (x, y, z) \end{cases}$$

$$\begin{cases} T = \dfrac{\vec{P}^2}{2m} \equiv -\dfrac{\hbar^2}{2m}\Delta \\ \\ V(\vec{x}) \end{cases}$$

$$E \equiv \hat{H}$$

Sometimes one writes also:

$$E \equiv i\hbar\frac{\partial}{\partial t} \qquad \left(i\hbar\frac{\partial}{\partial t}\psi = H\psi \right)$$

$$\psi(t,\vec{x}) = e^{-iEt} f(\vec{x}) \quad \to \quad i\hbar\frac{\partial}{\partial t}\psi(t) = E\,\psi(t,\vec{x})$$

How, achtung:

- t is a parameter, not an operator. it is on a completely $\neq$ level than $\vec{X}$.

(We make a meas. of $\hat{A}$ at t_0 = this is an inst. measurement!)

[illegible]

$$\hat{A} : L^2(-\infty,\infty) \mapsto L^2(-\infty,\infty)$$

$$\left(\psi(0,\vec{x}), \hat{A}\psi(0,\vec{x})\right) = \langle \hat{A} \rangle_{\psi(0,\vec{x})}$$

is the average of $\hat{A}$ if we have many equally prepared particles that we measure at time $t=0$.

Now, what can we say about $\langle \hat{A} \rangle_{\psi(t,\vec{x})}$? That is, at the instant t?

$$\langle \hat{A} \rangle_{\psi(t,\vec{x})} \equiv \langle \hat{A} \rangle(t) = \left(\psi(t,\vec{x}), \hat{A}\psi(t,\vec{x})\right)$$

$$\frac{d\langle \hat{A} \rangle}{dt} = \left(\frac{\partial \psi}{\partial t}, \hat{A}\psi\right) + \left(\psi, \hat{A}\frac{\partial \psi}{\partial t}\right)$$

$$= \left(-\frac{i}{\hbar}\hat{H}\psi, \hat{A}\psi\right) + \left(\psi, -\frac{i}{\hbar}\hat{A}\hat{H}\psi\right) =$$

$$i\hbar\frac{\partial \psi}{\partial t} = \hat{H}\psi \;\vdots$$

$$= \frac{i}{\hbar}\left(\hat{H}\psi, \hat{A}\psi\right) - \frac{i}{\hbar}\left(\psi, \hat{A}\hat{H}\psi\right) =$$

$$\overset{\hat{H}^{+}=\hat{H}}{=} \frac{i}{\hbar}\left(\psi, \hat{H}\hat{A}\psi\right) - \frac{i}{\hbar}\left(\psi, \hat{A}\hat{H}\psi\right) =$$

$$= \frac{i}{\hbar}\left(\psi, [\hat{H}, \hat{A}]\psi\right) = \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle_\psi .$$

Generalization for $\hat{A}(t)$ with an explicit t-dependence:

$$\frac{d\langle\hat{A}\rangle}{dt} = \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle_\psi$$

Analog. to classical evolution with Poisson - parentheses.

Ehrenfest theorem (1927):

averages in QM fulfill the classic e.o.m.:

1 dim:

- $$\frac{d\langle\hat{x}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle$$

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x}), \quad [\hat{H},\hat{x}] = \frac{1}{2m}[\hat{p}_x^2,\hat{x}] = -\frac{i\hbar}{m}\hat{p}_x$$

$$\boxed{\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}_x\rangle}{m}} \longleftrightarrow \boxed{p = mv}$$

- $$\frac{d\langle\hat{p}_x\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{p}_x]\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle$$

$$[\hat{H},\hat{p}_x] = [V(\hat{x}),\hat{p}_x] = i\hbar\frac{\partial V}{\partial x}$$

analog. to

$$\boxed{F = \frac{dp}{dt} = -\frac{\partial V}{\partial x}}$$

QM $\longleftrightarrow$ classical mechanics (only one aspect of it... it does not solve all the problems, but it is an important point).

$$i\hbar \frac{\partial \psi(t,\vec{x})}{\partial t} = \hat{H}\psi(t,\vec{x})$$

Ansatz:

$$\psi(t,\vec{x}) = \varphi(t)\, f(\vec{x})$$

$$i\hbar f(\vec{x}) \frac{d\varphi}{dt} = \varphi(t) \left(\hat{H} f(\vec{x}) \right)$$

$$i\hbar \frac{\frac{d\varphi}{dt}}{\varphi(t)} = \frac{\hat{H} f(\vec{x})}{f(\vec{x})} \equiv E \quad \text{(indep. on } t \text{ and } \vec{x}\text{)}$$

$$\underbrace{\hat{H} f(\vec{x}) = E f(\vec{x})}_{\text{stationary Schr. eq.}} \Rightarrow \text{Eigenvalues of } \hat{H}.$$

In some systems: E_n ($(n+\frac{1}{2})\hbar\omega$ for $V = \frac{1}{2} m\omega^2 x^2$)

In other cases E is cont.

In other cases it is a mixt.

$$E_n = -\frac{K}{n^2}, \quad E \geq 0$$

But then:

$$i\hbar \frac{\frac{d\varphi}{dt}}{\varphi(t)} = E \longmapsto \varphi(t) = e^{iEt/\hbar}$$

$\psi_E(t,\vec{x}) = e^{iEt/\hbar} f_E(\vec{x})$ is a solution of the time-dep. Schr. eq.

$E \longleftrightarrow i\hbar \frac{\partial}{\partial t}$

Sometimes one see:

$[E, t] = i\hbar$ (analog. to $[x, p] = i\hbar$)

How, CARE

t is a param. and not an operator. E corresponds to $\hat{H}$ and not to $i\hbar \frac{\partial}{\partial t}$ ..

$\Delta E \, \Delta t \geq \frac{\hbar}{2}$

also CARE: what is ΔE, what is Δt?
($\Delta E = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}$; Δt ?)

δt is, on the contrary, known.

$[\hat{H}, \hat{A}] \neq 0$

$\Delta \hat{H} \Delta \hat{A} \geq \frac{1}{2} |\langle i [\hat{H}, \hat{A}] \rangle|$

But:

$\frac{d \langle \hat{A} \rangle}{dt} = \frac{i}{\hbar} \langle [H, \hat{A}] \rangle$

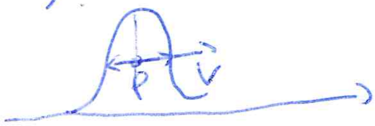
$$\Delta \hat{H} \cdot \Delta \hat{A} \geq \frac{\hbar}{2} \left| \frac{d\langle \hat{A} \rangle}{dt} \right|$$

if now we define: $\Delta t = \dfrac{\Delta \hat{A}}{\left| \dfrac{d\langle \hat{A} \rangle}{dt} \right|}$

we get

$$\Delta \hat{H} \, \Delta t \geq \frac{\hbar}{2} \qquad \Delta E \, \Delta t \geq \frac{\hbar}{2}.$$

(but this is related to the observable $\hat{A}$).

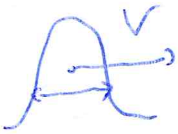


Note: $\hat{A} = \hat{x}$. $\Delta \hat{x} = l$

if I measure x

$$\Delta t = \frac{l}{v}.$$

if l is very small, Δt is very small, ΔE is very large.



Detector... it makes click (with $\delta t \approx 0$) at a certain time when the particle goes through it.

if l is very large, Δt is very large

$$\Delta t = \sqrt{\langle t^2 \rangle - \langle t \rangle^2} \quad (\text{not } \delta t)$$

but ΔE is then very small.

(Again, all I mean ΔE or Δt (bzw x))

Unstable state:

$$P(t) = e^{-\Gamma t}$$

$$\tau \sim \hbar/\Gamma \sim \hbar/\Delta E$$

$$\Gamma = 2\Delta E$$

How, it decays... the smaller ΔE, the longer it lives and viceversa.