Operators and Commutation relations

$L^2(a,b)$

$f(x): \mathbb{R} \mapsto \mathbb{C} \in L^2(a,b)$ if $\int_a^b |f(x)|^2 dx < \infty$

Operators:

$\hat{A}: L^2(a,b) \mapsto L^2(a,b)$

$f \mapsto \hat{A} f = g \in L^2$

Linear operators:

$\hat{A}(c_1 f_1 + c_2 f_2) = c_1(\hat{A} f_1) + c_2(\hat{A} f_2)$

Example of operators:

$\hat{x}: f(x) \mapsto x f(x)$ (Hermitian)

$\left\{ \begin{array}{l} \hat{\dfrac{d}{dx}}: f(x) \mapsto \dfrac{df(x)}{dx} \quad \text{(Antihermitian)} \\ \hat{P}_x = -i\hbar \dfrac{d}{dx} \end{array} \right.$

One studies in QM the eigenvalue equations:

$$\hat{A}\,\varphi_i = \lambda_i\,\varphi_i \qquad f(x) = \sum_i c_i \varphi_i$$

For the momentum operator:

$$\hat{P}\,\varphi_k = P\,\varphi_k \qquad p = \hbar k$$

$$\varphi_k = \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$\hat{P}\varphi_k = -i\hbar \frac{d\varphi_k}{dx} = (\hbar k)\,\varphi_k = P\,\varphi_k$$

φ_k describes a particle with momentum $\hbar k$ (speed $\frac{\hbar k}{m} = v$)

[In 3d: $\varphi_{\vec{k}} = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \Rightarrow \vec{p} = \hbar\vec{k}$]

Then:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, g(k)\, e^{ikx}$$

$\Rightarrow$ superposition of f(x) in the basis of eigenfun of $\vec{p}$

mathematics: Fourier Transform

$\int_{k_1}^{k_2} |g(k)|^2\, dk$ → prob. to find p in the range $\hbar k_1$ and $\hbar k_2$.

$\hat{x}$ operator:

$$\hat{x}\,\phi(x) = x\,\phi_{x_0}(x) = x_0\,\phi_{x_0}(x)$$

$$\phi_{x_0}(x) \equiv \phi_{x_0}(x) = \delta(x-x_0)$$ ⇒ 1 state with definite position (although "Achtung": NO norm.)

$$\underbrace{f(x) = \int_{-\infty}^{\infty} dx_0\, \phi_{x_0}(x)\, f(x_0)}_{} = \int_{-\infty}^{\infty} dx_0\, \delta(x-x_0)\, f(x_0) = f(x)$$

f(x) written as a superposition of eigenstates of the position.

$\int_{x_1}^{x_2} |f(x)|^2 dx$ is the prob. to find the particle between x_1 and x_2.

Summary:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, g(k)\, e^{ikx} = \int_{-\infty}^{\infty} dx_0\, f(x_0)\, \delta(x-x_0)$$

eigenvalue of $\hat{p}$
$p = \hbar k$

eigenvalue of $\hat{x}$: x_0.

In $L^2(a,b)$ there is a scalar product:

$$(\varphi_1, \varphi_2) = \int_a^b \varphi_1^* \varphi_2 \, dx \quad \in \mathbb{C}$$

Given the operator $\hat{A}$, the conjugate $\hat{A}^+$ is defined by:

$$(\varphi_1, \hat{A}\varphi_2) = (\hat{A}^+ \varphi_1, \varphi_2)$$

$$= \int_a^b \varphi_1^* (\hat{A}\varphi_2) \, dx = \int_a^b (\hat{A}^+ \varphi_1)^* \varphi_2 \, dx$$

An operator is Hermitian if $\underline{\hat{A} = \hat{A}^+}$.

$$(\varphi_1, \hat{A}\varphi_2) = (\hat{A}\varphi_1, \varphi_2) \qquad \forall \varphi_1, \varphi_2 \in L^2(a,b).$$

[$\hat{x}$ and $\hat{p} = -i\hbar \frac{d}{dx}$ are both hermitian!]

Remind also:

$\frac{d}{dx}$ is anti-hermitian: $\left(\frac{d}{dx}\right)^+ = -\frac{d}{dx}$ (assuming periodic function at the borders)

Simple properties and defs:

(i) $\hat{A}$ generic operator: $L^2(a,b) \mapsto L^2(a,b)$

(ii) $\hat{S} = \hat{A} + \hat{A}^\dagger$ is Hermitian.

$$\left[\hat{S}^\dagger = \hat{A}^\dagger + \hat{A}^{\dagger\dagger} = \hat{A}^\dagger + \hat{A} = \hat{A} + \hat{A}^\dagger = \hat{S} \right]$$

$\hat{D} = \hat{A} - \hat{A}^\dagger$ is Anti-Hermitian.

$$\left[\hat{D}^\dagger = \hat{A}^\dagger - \hat{A}^{\dagger\dagger} = \hat{A}^\dagger - \hat{A} = -(\hat{A} - \hat{A}^\dagger) = -\hat{D}^\dagger \right]$$

(iii) $\hat{A}, \hat{B}$ operators.

$$(\hat{A}\hat{B})f(x) = \hat{A}(\hat{B}f(x))$$

$$\hat{P} = \hat{A}\hat{B}$$

$$\hat{P}^\dagger = (\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \quad !$$

if $\hat{A}$ and $\hat{B}$ are hermitian, $\hat{P}$ is not necessary such! Namely: $\hat{P}^\dagger = \hat{B}\hat{A}$
$\neq \hat{A}\hat{B}$

(they are equal only if $\hat{A}$ and $\hat{B}$ commute!)

(iv) $\hat{A}$ herm. | $\hat{A}$ antiherm.
$\hat{B} = i\hat{A}$ is antiherm. | $\hat{B} = i\hat{A}$ is herm.

Commutator:

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Properties:

1) $\hat{A}^{+} = \hat{A}$, $\hat{B}^{+} = \hat{B}$

$[\hat{A},\hat{B}]$ is anti-hermitian: $\left([\hat{A},\hat{B}] = i\hat{C} \text{ with } \hat{C}^{+} = \hat{C} \right)$

$$[\hat{A},\hat{B}]^{+} = (\hat{A}\hat{B} - \hat{B}\hat{A})^{+} = \hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+} = \hat{B}\hat{A} - \hat{A}\hat{B} = -[\hat{A},\hat{B}].$$

2) $[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$

3) $[\hat{A},\hat{B}+\hat{C}] = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$

4) $[\hat{A},\hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$

$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

5) $[\hat{A},[\hat{B},\hat{C}]] + [\hat{C},[\hat{A},\hat{B}]] + [\hat{B},[\hat{C},\hat{A}]] = 0$.

"crucial commutator"

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

Let us evaluate:

$$[\hat{x},\hat{p}](f(x)) = \left(x\left(-i\hbar\frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)x\right)f(x) =$$

$$= -i\hbar x\frac{df}{dx} + i\hbar\frac{d}{dx}\left(x f(x)\right) =$$

$$= \cancel{-i\hbar x\frac{df}{dx}} + i\hbar f(x) + \cancel{i\hbar x\frac{df}{dx}}$$

$$= i\hbar f(x) \qquad \forall f(x)$$

Ergo:

$$[\hat{x},\hat{p}] = i\hbar$$

Both $\hat{x},\hat{p}$ are herm. $\rightarrow [\hat{x},\hat{p}] = i(\underbrace{\hbar 1}_{c\ \text{herm. operator}})$

- Remember the Poisson parentheses? $\{x,p\}_p = 1$!!! Formally: Poisson $\downarrow$ commutator

0

Going further:

$[x,p] = i\hbar$

calculate:

- $[x^2,p] = x\overset{+i\hbar}{[x,p]} + \overset{+i\hbar}{[x,p]}x = 2i\hbar x$; $[x^n,p] = i\hbar(nx^{n-1})$

- $[h(x),p] = i\hbar \frac{dh}{dx}$

$[x, g(p)] = i\hbar \frac{dg}{dp}$

$=$ $H = T + V = \frac{p^2}{2m} + V(x)$

Do x and p commute with H? In general no (Remind: commutation with H means constant of motion)

$[H,x] = \left[\frac{p^2}{2m}, x\right] = -i\hbar \frac{p}{m} \neq 0$

$[H,p] = [V(x),p] = i\hbar \frac{dV}{dx} \neq 0$

This means also: eigenstate of H are <u>not</u> eigenstates of P and x.

What is the operator $\hat{x}\hat{p}$? Is it hermitian?

$$(xp)^{+} = p^{+}x^{+} = px = px - xp + xp = [p,x] + xp$$

$$= -i\hbar + xp$$

Ergo:

$$(xp)^{+} = (xp) - i\hbar \neq (xp)$$

The operator xp is not hermitian... note, classically, we have no problem in discussing position x momentum... but in QM there is an issue when constructing this quantity.

Which is the analogous operator in QM? It must be Hermitian:

$$\frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$$

is the correct QM-counterpart of xp.

Indetermination relation

We have two observables $\hat{A}, \hat{B}$ $(\hat{A}^+ = \hat{A}\,;\ \hat{B}^+ = \hat{B})$.

The sentence "can we measure both of them at the same time" is actually meaningless.

We can measure first $\hat{A}$ and just a bit later $\hat{B}$ (or viceversa).

Now, if $[\hat{A}, \hat{B}] = 0 \Leftrightarrow$ I can construct eigenfunctions φ_n which are at the same time eigenvalues of $\hat{A}$ and $\hat{B}$:

$$\begin{cases} \hat{A}\varphi_n = a_n \varphi_n \\ \hat{B}\varphi_n = b_n \varphi_n \end{cases}$$

$\Leftarrow$

$$[A,B]\varphi_n = (a_n b_n - b_n a_n)\varphi_n = 0 \quad \forall n \to [A,B] = 0$$

$\Rightarrow$

Let us consider $\varphi_n \,/\, \hat{A}\varphi_n = a_n \varphi_n$

$$[A,B]\varphi_n = A(B\varphi_n) - B(A\varphi_n) = A(B\varphi_n) - B\, a_n \varphi_n = 0$$

$$\to A(B\varphi_n) = a_n (B\varphi_n)$$

$B\varphi_n$ is eigenfunction of A with eigenvalue a_n.

if f_n is not degenerate:

$$B\varphi_n = q_n \varphi_n \quad \Rightarrow \varphi_n \text{ is also eigenf. of } B$$ (with a new eigenv. denoted as q_n.

The proof for the most general case (degeneration is possible) is much more complicated.

$$A\varphi_1 = a\varphi_1$$
$$A\varphi_2 = a\varphi_2$$

then you get that $B\varphi_1$ and $B\varphi_2$ are also eigenfunctions of A.

$$B\varphi_1 = \alpha\varphi_1 + \beta\varphi_2$$
$$B\varphi_2 = \gamma\varphi_1 + \delta\varphi_2$$

it is now possible to write down two $\perp$ combinations $\psi_1 = \# \varphi_1 + \# \varphi_2$

$\psi_2 = -\# \varphi_1 + \# \varphi_2$

(as in:
$\psi_1 = c\varphi_1 + s\varphi_2$
$\psi_2 = -s\varphi_1 + c\varphi_2$)

$$B\psi_1 = q_1 \psi_1$$
$$B\psi_2 = q_2 \psi_2$$

(while both ψ_1, ψ_2 are both eigenvect. of A with eigenvalue a)

Suppose now that we start from

$$\psi(0,x) = \sum_n c_n \varphi_n$$

No degeneracy:

If we make a measurement of the observable $\hat{A}$,

the result is a_n with prob. $|c_n|^2$.

The wave function collapses to

$$\varphi_n$$

If then I measure B, I find b_n.

(Note, if I had done the opposite → same result).

<u>Degeneracy</u>: $\varphi_n, \varphi_{n+1}, \dots \varphi_{n+m}$ ⇒ SAME EIGENVALUE a_n

we find $|a_n|^2$ with prob. $|c_n|^2 + |c_{n+1}|^2 + \dots |c_{n+m}|^2$

The wave function collapses to:

$$\psi(0^+,x) = N\left(c_n \varphi_n + c_{n+1}\varphi_{n+1} + \dots + c_{n+m}\varphi_{n+m}\right)$$

$$N = \frac{1}{\sqrt{|c_n|^2 + \dots + |c_{n+m}|^2}}$$

Now, each of these is also an eigenf. of B.

Ergo, if now we measure B we find (no deg. for B): b_n with prob.

$|N|^2|C_n|^2$ → collapse to φ_n only

(a degeneracy in B → prob. $|N|^2(|C_n|^2+|C_{n+1}|^2+\dots)$).

Then, the prob. to find a_n, b_n is

$$\left(|C_n|^2+\dots|C_{n+m}|^2\right)\cdot|N|^2\cdot|C_n|^2=|C_n|^2.$$

Viceversa, if I had measured first B and then A → same result, ~~the~~ same outcomes.

"One other ways: we can measure them at the same time... ACTIONS".

φ_1	φ_2	φ_3	φ_4	φ_5	a_1
φ_6					$a_6 \neq a_1$
$\boxed{\varphi_7}$					
b_1					

$A\varphi_i = a_1 \varphi_i \qquad i = 1, 2, \dots 5$

$$\begin{cases} B\varphi_1 = b_1\varphi_1 \\ B\varphi_6 = b_1\varphi_1 \\ B\varphi_7 = b_1\varphi_7 \end{cases}$$

$A\varphi_2 = b_2\varphi_1$ with $b_2 \neq b_1 \dots \quad (b_1 = b_6 = b_7)$

$$\psi(0^-, x) = \sum_n c_n \varphi_n$$

Measure $\hat{A}$:

The prob. to measure a_1 is $\quad |c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 + |c_5|^2$.

The state collapse to

$$\psi(0^+, x) = N \sum_{n=1}^{5} c_n \varphi_n \qquad N = \frac{1}{\sqrt{|c_1|^2 + \dots |c_5|^2}}$$

Then, let us do a meas. of B. The prob. to find b_1 is

$$|N|^2 \cdot (|c_1|^2)$$

Then, the prob. to measure $(a_1 \wedge b_1)$ is:

$$(|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 + |c_5|^2) \cdot |N|^2 \cdot |c_1|^2 = |c_1|^2 \, !$$

The wf after both meas is

$$\psi(0^{++}, x) = \varphi_1(x)$$

(with prob. $|c_1|^2$).

If now we repeat a meas. of A we find a_1 certainly as we know and if we meas. B we find b_1 certainly as we want, but we should do these meas. right after the first one, otherwise the time evolution can bring us far away from that.

Now, if we first perform a meas. of B:

$$|c_1|^2 + |c_6|^2 + |c_7|^2 \to b_1$$

$$\psi' = N' (c_1\varphi_1 + c_6\varphi_6 + c_7\varphi_7) \qquad N' = \frac{1}{\sqrt{|c_1|^2+|c_6|^2+|c_7|^2}}$$

Now we meas. A $\to$ the prob. to find a_1 is

$$|N'|^2 \cdot |c_1|^2$$

Summing up $\to$ a_1, b_1 with prob.

$$(|c_1|^2 + |c_6|^2 + |c_7|^2) \cdot |N'|^2 |c_1|^2 = |c_1|^2$$

After both meas. $\to$ prob. is $|c_1|^2$ just as before, $\psi(0^{++}, x) = \varphi_1$!

A more complicated example:

$$\varphi_1 \quad \varphi_2 \quad \varphi_3 \qquad a_1$$

$$\varphi_4$$

$$\sim$$

$$b_1$$

$$\psi = \sum_{n=1}^{\infty} c_n \varphi_n$$

Meas. $A \to a_1$ with prob. $|c_1|^2 + |c_2|^2 + |c_3|^2$

$$\psi = N(c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3).$$

Meas. B ↓
Now, prob. to find b_1 is $|N|^2 \cdot (|c_1|^2 + |c_2|^2)$

The prob. to find a_1 first and b_1 after is:

$$(|c_1|^2 + |c_2|^2 + |c_3|^2)\,|N|^2\,(|c_1|^2 + |c_2|^2) = |c_1|^2 + |c_2|^2$$

Then, the wf is:

$$\psi = \frac{1}{\sqrt{|c_1|^2 + |c_2|^2}}(c_1\varphi_1 + c_2\varphi_2)$$

By first meas. B and then $A \mapsto$ same result.

A very interesting case takes place when two observables do NOT commute:

$$[\hat{A},\hat{B}] \neq 0.$$

In this case there is no common set of eigenfunctions.

$$\begin{cases} \hat{A}\varphi_n = a_n \varphi_n \\ \hat{B}\eta_k = b_k \eta_k \end{cases} \qquad \begin{cases} (\varphi_n, \varphi_m) = \delta_{nm} \\ (\eta_k, \eta_r) = \delta_{kr} \end{cases}$$

$$\psi(0,x) = \sum_{n=1}^{\infty} c_n \varphi_n = \sum_{k=1}^{\infty} d_k \eta_k \qquad \sum_{n=1}^{\infty} |c_n|^2 = \sum_{k=1}^{\infty} |d_k|^2 = 1$$

Equivalent writing of $\psi(0,x)$.

Meas $\hat{A}$:

The prob. to find $|a_1|$ is $|c_1|^2$ (no degeneracy for simplicity).

But then, the wf of the state after this first meas. is:

$$\psi(0^+,x) = \varphi_1(x).$$

But now φ_1 is NOT an eigenstate of B.

Let us write $\varphi_1 = \sum_{k=1}^{\infty} \alpha_{1k} \eta_k$

Then, if I meas. B I find b_1 with prob.

$$|\alpha_{11}|^2$$

Then, the w.f. after the 2nd meas. is: η_1.

The prob. for $(a_1 \wedge b_1)$ is:

$$|c_1|^2 \cdot |\alpha_{11}|^2$$

Now, let us reverse the order:

We first measure B; the prob. to find b_1 is $|d_1|^2$. Then we have η_1, which

$$\eta_1 = \sum_{n=1}^{\infty} \beta_{1n} \varphi_n$$

out of which we get, by a meas. of $\hat{A}$ the result a_1 with prob. $|\beta_{11}|^2$.

Put it together:

$$|\beta_{11}|^2 \cdot |c_1|^2$$

and the wf is φ_1.

$$\psi = \sum_n c_n \varphi_n = \sum_k d_k \eta_k$$

A $\quad |c_1|^2 \to \varphi_1$	$|d_1|^2 \mapsto \eta_1$ $\quad$ B
$\varphi_1 = \sum_k \alpha_{1k} \eta_k$	$\eta_1 = \sum_n \beta_{1n} \varphi_n$
B $\quad |\alpha_{11}|^2 \to \eta_1$	$|\beta_{11}|^2 \mapsto \varphi_1$ $\quad$ A
Total prob: $|c_1|^2 \cdot |\alpha_{11}|^2$	Total prob. $|d_1|^2 \cdot |\beta_{11}|^2$
$\psi(0^{++}, x) = \eta_1$	$\psi(0^{++}, x) = \varphi_1$
$\downarrow$ if I measure B I find b_1... but if I measure A I get (possibly) something else!!!	if I measure A $\to$ I find a_1 but if I measure B I get $b_1, b_2, \ldots$ many possible $\neq$ results!!!

Example:

φ_1, φ_2 $\qquad$ η_1, η_2

$$\begin{cases} \varphi_1 = \cos\theta\, \eta_1 + \sin\theta\, \eta_2 \\ \varphi_2 = -\sin\theta\, \eta_1 + \cos\theta\, \eta_2 \end{cases}$$

$$\begin{cases} \eta_1 = \cos\theta\, \varphi_1 - \sin\theta\, \varphi_2 \\ \eta_2 = \sin\theta\, \varphi_1 + \cos\theta\, \varphi_2 \end{cases}$$

$$\boxed{\psi = \sqrt{\frac{2}{3}}\, \varphi_1 + \sqrt{\frac{1}{3}}\, \varphi_2}$$

$$\begin{cases} A\varphi_1 = a_1 \varphi_1 \\ A\varphi_2 = a_2 \varphi_2 \end{cases}$$

$$\begin{cases} B\varphi_1 = b_1 \varphi_1 \\ B\varphi_2 = b_2 \varphi_2 \end{cases}$$

$$\psi = \sqrt{\frac{2}{3}}\left(c\eta_1 + s\eta_2\right) + \sqrt{\frac{1}{3}}\left(-s\eta_1 + c\eta_2\right) =$$

$$\boxed{\psi = \left(\sqrt{\frac{2}{3}}\, c - \sqrt{\frac{1}{3}}\, s\right)\eta_1 + \left(\sqrt{\frac{2}{3}}\, s + \sqrt{\frac{1}{3}}\, c\right)\eta_2}$$

Measure A and B

Prob. $\frac{2}{3} \to a_1$; $\varphi_1 = c\,\eta_1 + s\,\eta_2$

$a_1 \wedge b_1 \to \frac{2}{3}\,(\cos^2\theta)\,? \to \psi = \eta_1$

Measure B and A.

Prob: $\left(\sqrt{\tfrac{2}{3}}\,c - \sqrt{\tfrac{1}{3}}\,s\right)^2 \to \eta_1 = c\,\varphi_1 - s\,\varphi_2$

Exp: $\left(\sqrt{\tfrac{2}{3}}\,c - \sqrt{\tfrac{1}{3}}\,s\right)^2 \cdot c^2$ with $\psi = \varphi_1$ ... in this case...

$\neq \frac{2}{3}\,c^2$ (since $\theta \neq 0$ ... but $\theta = 0$ means that they are eigenvalues).

General formula

$\hat{A}, \hat{B} \ / \ [\hat{A},\hat{B}] \neq 0.$

$\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$; $\langle\hat{A}\rangle = (\psi, \hat{A}\psi\rangle$

$\langle\hat{A}^2\rangle = (\psi, \hat{A}^2\psi)$

$\Delta\hat{B} = \sqrt{\langle\hat{B}^2\rangle - \langle\hat{B}\rangle^2}$;

$$\boxed{\Delta\hat{A}\cdot\Delta\hat{B} \geq \frac{1}{2}\left|\langle i[\hat{A},\hat{B}]\rangle\right|} \qquad (*)$$

[$\Delta\hat{A} = 0$ only if $\hat{A}\psi = a\psi$

For $[\hat{x},\hat{p}] = i\hbar$

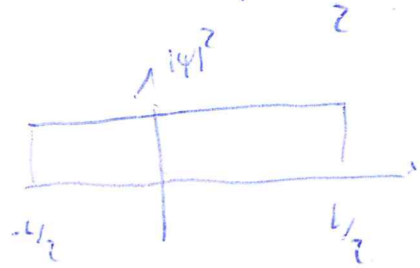
$\Delta\hat{x}\,\Delta\hat{p} \geq \frac{\hbar}{2}$ (we saw it with the Gauss distribution one week ago).

But conceptually there is a bit different. Here, for ΔA and ΔB, we do not need to measure A and B on the same particle ... better: we should not. The meaning is that: we prepare the same particle 10000 times, on the 50% we meas. A, or the other 50% B. Then, $\Delta\hat{A}\Delta\hat{B}$ fulfills (*).

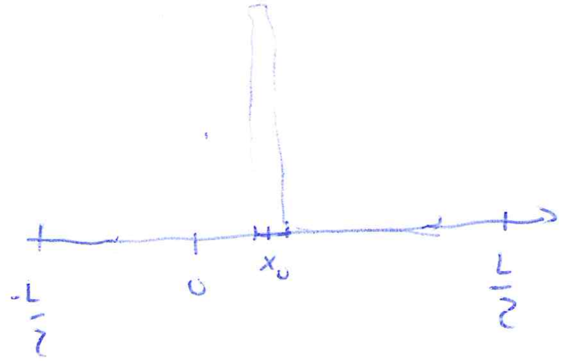
In the "Gaussian" wf → ... $\Delta x \, \Delta p = \frac{\hbar}{2}$

but in general $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$

Measure x and p on the same particle:



$$\psi = N e^{+ i k_0 x}$$



Measure $p \to \hbar k_0$, ψ doesn't change.

Measure $x \mapsto x_0$ with prob. $\frac{\delta x}{L}$

The wf after the x-meas. is:

$$\tilde{N} \, \delta(x - x_0) \propto \int dk \, e^{ikx}$$

if I measure p again ... I get with equal prob. each result for p ... Prob. $p_0 \wedge x_0 = \frac{\delta x}{L}$

If you first measure x you get $\delta(x - x_0)$ with prob. $\frac{\delta x}{L}$.

Then, meas. $p \mapsto p_0$ with prob. $\frac{\delta p}{L}$.

Total prob. $= \frac{\delta x}{L} \cdot \frac{\delta p}{L}$.