Plan of lecture 5

Fourier transf. and operators $\hat{x}, \hat{p}_x$.

- $L^2(-\pi,\pi)$, basis $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}$. Examples.

- Extension to $L^2(-\lambda\pi, \lambda\pi)$ and then $\lambda \mapsto \infty$ = Fourier transform.

  Parseval

  Link to δ

  3 important examples.

- Gaussian w f and its physical meaning. $\Delta x, \Delta p$

- Eigenfunctions of $\hat{p}_x$

- " " " " $\hat{x}$.

$L^2(-\pi,\pi)$

A ON basis is: $\left\{ \varphi_n = \frac{e^{inx}}{\sqrt{2\pi}} \right\}$ $\rightarrow$ Each $f(x) \in L^2(-\pi,\pi)$ / $f(x) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(x)$

$$\frac{e^{inx}}{\sqrt{2\pi}} = \frac{\cos(nx)}{\sqrt{2\pi}} + i\,\frac{\sin(nx)}{\sqrt{2\pi}}.$$

Nb: the functions of the basis are periodic. $\frac{e^{-in\pi}}{\sqrt{2\pi}} = \frac{e^{in\pi}}{\sqrt{2\pi}}$

Namely, for $n = 0, \pm 2, \pm 4, \dots$ $\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$

$n = \pm 1, \pm 3, \dots$ $\frac{-1}{\sqrt{2\pi}} = -\frac{1}{\sqrt{2\pi}}$

$$(\varphi_n, \varphi_m) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \frac{e^{imx}}{\sqrt{2\pi}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

$$\frac{1}{2\pi}\frac{1}{i(m-n)}\left(e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right) = 0!!!$$

Note: $\varphi_n = \frac{e^{inx}}{\sqrt{2\pi}}$ is an eigenfunction of $\hat{P}_x = -i\hbar\partial_x$.

Namely:

$$\hat{P}_x \varphi_n = -i\hbar \frac{\partial}{\partial x}\left(\frac{e^{inx}}{\sqrt{2\pi}}\right) = \hbar n \left(\frac{e^{inx}}{\sqrt{2\pi}}\right) = \hbar n\, \varphi_n$$

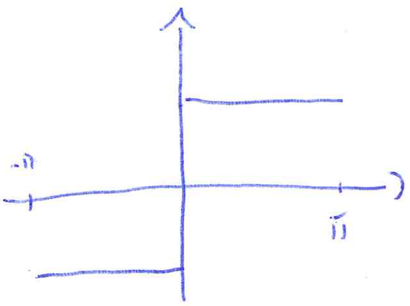
$$f = \sum_{n=-\infty}^{\infty} c_n \varphi_n$$

$$\boxed{(f,f) \overset{\text{Parseval}}{=} \sum_{n=-\infty}^{\infty} |c_n|^2}$$

<u>Note:</u> although the basis is periodic, I can approximate each L^2-integrable function (even non-periodic ones!

$$\mathcal{E}(x) = \begin{cases} -1 & x \in (-\pi, 0) \\ 1 & x \in (0, \pi) \end{cases}$$

is obviously non-periodic (and also not cont.)



$$\mathcal{E}(x) = \sum_{n=-\infty}^{\infty} c_n \varphi_n$$

$$c_n = (\varphi_n, \mathcal{E}) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \mathcal{E}(x)\,dx = \int_{-\pi}^{0} \frac{e^{-inx}}{\sqrt{2\pi}} \overbrace{\mathcal{E}(x)}^{-1}\,dx + \int_{0}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \overbrace{\mathcal{E}(x)}^{+1}\,dx$$

$$= -\int_{-\pi}^{0} \frac{e^{-inx}}{\sqrt{2\pi}}\,dx + \int_{0}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}}\,dx =$$

$$\overset{n \neq 0}{=} -\frac{1}{(-in)} \left(\frac{e^{-inx}}{\sqrt{2\pi}} \right)\Bigg|_{-\pi}^{0} + \frac{1}{(-in)} \left(\frac{e^{-inx}}{\sqrt{2\pi}} \right)\Bigg|_{0}^{\pi} =$$

$$= \frac{1}{in\sqrt{2\pi}} \left(1 - e^{in\pi} \right) - \frac{1}{in\sqrt{2\pi}} \left(e^{-in\pi} - 1 \right)$$

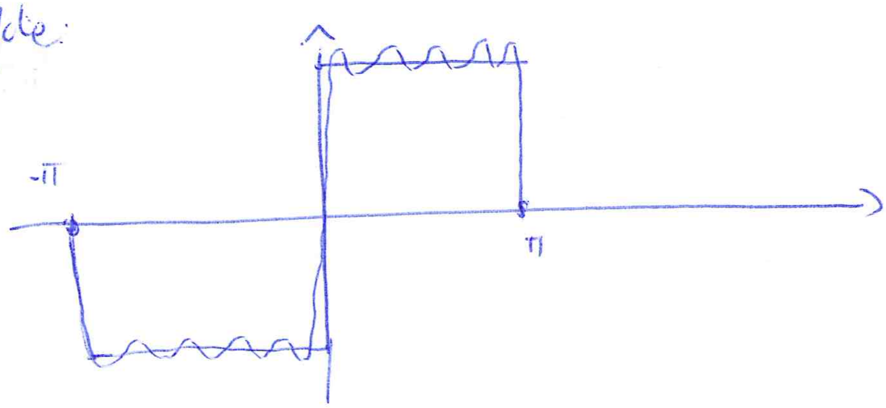
$$= \frac{2}{im\sqrt{2\pi}} - \frac{1}{im\sqrt{2\pi}}\left(e^{im\pi} + e^{-im\pi}\right) =$$

$$= \frac{2}{im\sqrt{2\pi}} - \frac{1}{im\sqrt{2\pi}}\left(2\cos(m\pi)\right)$$

$$= i\,\frac{2\cos(m\pi)}{m\sqrt{2\pi}} - i\,\frac{2}{m\sqrt{2\pi}} = \frac{i\,2}{m\sqrt{2\pi}}\left((-1)^m - 1\right) \quad \text{for } m \neq 0$$

$$\boxed{\begin{aligned} c_0 &= -\int_{-\pi}^{0} \frac{1}{\sqrt{2\pi}}\,dx + \int_{0}^{\pi} \frac{1}{\sqrt{2\pi}}\,dx \\ &= -\frac{1}{\sqrt{2\pi}}\left(0 - (-\pi)\right) + \frac{1}{\sqrt{2\pi}}\cdot(\pi) = 0! \end{aligned}}$$

Note:



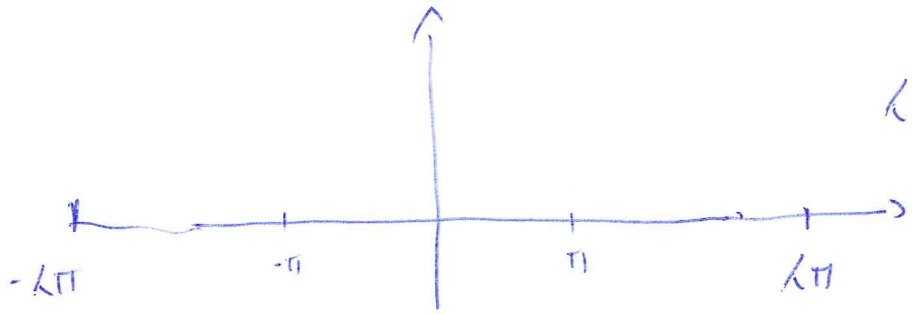
The points $x = \pm\pi, 0$ cannot be actually punctually described!!!!

Extension to $L^2(-\lambda\pi, \lambda\pi)$:

$$\left\{ \frac{e^{inx/\lambda}}{\sqrt{2\pi\lambda}} \right\}$$ is a ON basis in $(-\lambda\pi, \lambda\pi)$

$$(\varphi_n, \varphi_n) = \int_{-\lambda\pi}^{\lambda\pi} \frac{e^{-inx/\lambda}}{\sqrt{2\pi\lambda}} \frac{e^{inx/\lambda}}{\sqrt{2\pi\lambda}} \, dx = 1 \,!$$

$$f(x) = \sum_n c_n \varphi_n(x), \qquad \boxed{c_n = (\varphi_n, f) = \int_{-\pi\lambda}^{\pi\lambda} \frac{e^{-inx/\lambda}}{\sqrt{2\pi\lambda}} f(x) \cdot dx}$$

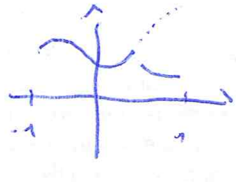


$\lambda > 1$

Now, the "trick" is to send $\lambda \mapsto \infty$!!!

Answer to questions:

1) $\{x^n\}$ $n=0,1,2,\ldots$ is a complete basis in the segment $(-1,1)$.



$$f(x)=\sum_{n=0}^{\infty} c_n x^n$$

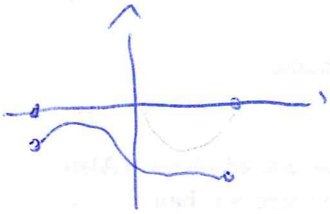
But ACHTUNG: only if $f(x)$ is C^∞, then $c_n = \frac{1}{n!} f^{(n)}(0)$! otherwise the determination of c_n is much more complicated.
(In this case the fact that the basis is not $\perp$ makes the det. of c_n more difficult... theory of $\perp$ polinomials....)

2)

$$\left(\varphi_i, \frac{d}{dx}\varphi_j\right) = \left(\varphi_i^* \varphi_j\right)_a^b - \int_a^b \frac{d\varphi_i^*}{dx}\varphi_j \, dx$$

if the basis is periodic $\to 0 \to$ Antih.

$$\neq \quad (f_i, g) = \underbrace{(f^* g)_a^b}_{\neq 0} - \int_a^b \frac{df^*}{dx} g \, dx$$



$\Rightarrow$ but if you express it in term of the basis you get "0"... it is like that because $f(x)=\sum_i c_i\varphi_i$ doesn't mean a point-by-point convergence!

$L^2(-\infty, \infty)$

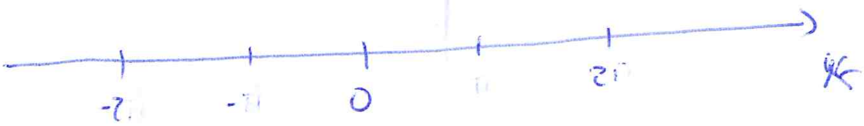
As usual, when sending a parameter to ∞ care is needed....

first, we introduce the "cont. parameter" k as

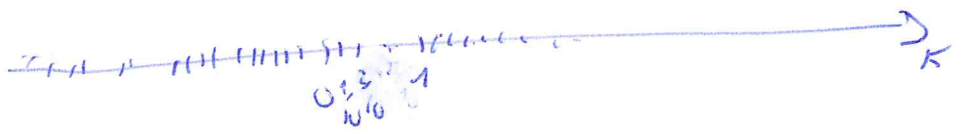
$$k = \frac{n}{\lambda}$$

if λ is very large, k is (for a finite λ) also discrete.... but becomes more densely packed.

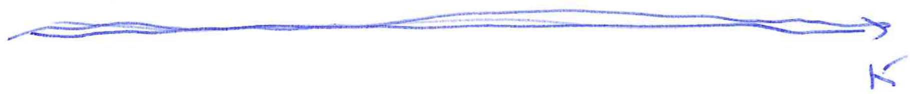
$\lambda = 1$



$\lambda = 10$



$\lambda \to \infty$ $k \to$ cont. parameter.



$$c_n = \int_{-\pi\lambda}^{\pi\lambda} \frac{e^{-in\,x/\lambda}}{\sqrt{2\pi\lambda}} f(x)\,dx$$ for $\lambda \mapsto \infty \to$ goes to zero...

but we define:

$$g(k) = \lim_{\lambda \mapsto \infty} \sqrt{\lambda}\, c_n = \lim_{\lambda \mapsto \infty} \sqrt{\lambda} \int_{-\pi\lambda}^{\pi\lambda} \frac{e^{-in\,x/\lambda}}{\sqrt{2\pi\lambda}} f(x)\,dx$$

$$\boxed{g(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)\,dx}$$

Thus is [illegible] the Fourier transform of $f(x)$.

$$c_n = \begin{pmatrix} \vdots \\ c_{10} \\ \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \end{pmatrix} \longmapsto g(k)$$

$(\lambda \mapsto \infty)$
becomes cont.
(überabzählbar)

$$\left\{ \frac{e^{-ikx}}{\sqrt{2\pi}} \right\}$$ is a ON basis for $L^2(-\infty, \infty)$
↳ generalized, see later

Parseval:

$$(f,f) = \int_{-\pi\lambda}^{\pi\lambda} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 =$$

$$= \sum_{n=-\infty}^{\infty} \left| \int_{-\pi\lambda}^{\pi\lambda} \frac{e^{-inx/\lambda}}{\sqrt{2\pi\lambda}} f(x)\, dx \right|^2 =$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{\lambda}}_{dk} \left| \underbrace{\int_{-\pi\lambda}^{\pi\lambda} \frac{e^{-inx/\lambda}}{\sqrt{2\pi\lambda}} f(x)\, dx}_{g(k)} \right|^2$$

$n \mapsto k$
$\frac{1}{\lambda} \mapsto dk$

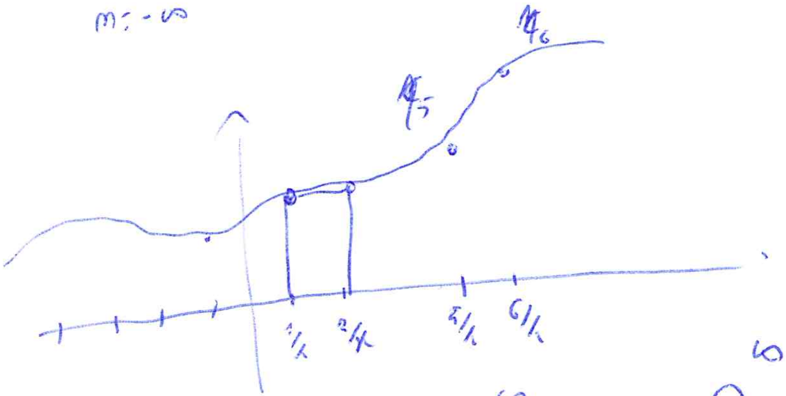
$$= \int_{-\infty}^{\infty} dk\, |g(k)|^2$$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\phi f(k)|^2 dk}$$

gen. Parseval for Fourier transf.

N.b. = From Σ to $\int$

$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda} \cdot \eta_n$$



For λ very large ...
$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda} \equiv \int_{-\infty}^{\infty} dk \, \eta_{n=\lambda k}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx/\lambda}}{\sqrt{2\pi\lambda}} = \sum_{n=-\infty}^{\infty} \frac{g(k)}{\sqrt{\lambda}} \frac{e^{inx/\lambda}}{\sqrt{2\pi}\cdot\sqrt{\lambda}} =$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} g(k) \frac{e^{inx/\lambda}}{\sqrt{2\pi}} \overset{\lambda\to\infty}{=} \int_{-\infty}^{\infty} \frac{g(k) e^{ikx}}{\sqrt{2\pi}} dk$$

Put them together:

$$\begin{cases} g(k) = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-ikx} \\ f(x) = \displaystyle\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} g(x) e^{ikx} \end{cases}$$

Fully symmetrical :)

Fourier transf. and Dirac δ

Plug the 1° eq. into the second:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \underbrace{\left(\int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}} f(x') e^{-ikx'} \right)}_{g(k)} e^{ikx}$$

$$f(x) = \int_{-\infty}^{\infty} dx' \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \right] f(x')$$

Ergo:

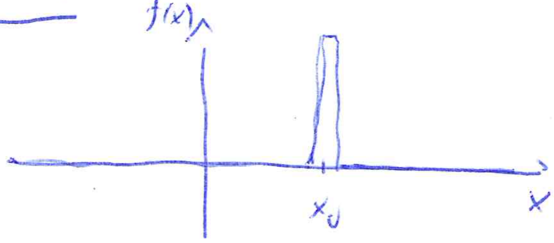
$$\boxed{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x')}$$

We indeed already ~~had~~ this representation of the Dirac-δ.

ON ~~ON~~ of the basis $\left\{ \frac{e^{ikx}}{\sqrt{2\pi}} \right\}$: the concept of norm. is extended.

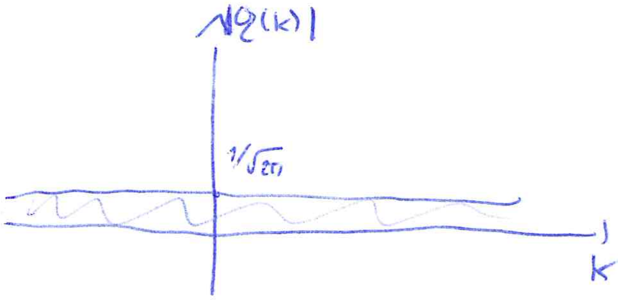
Two important mathematical limits

$$f(x) = \delta(x - x_0)$$



$$g(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, \delta(x-x_0)\, e^{-ikx} = \frac{e^{-ikx_0}}{\sqrt{2\pi}}$$

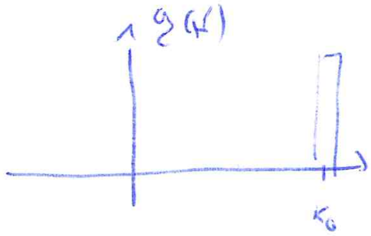
$$|g(k)|^2 = \frac{1}{\sqrt{2\pi}}$$



Completely "delocalised".

=

$$g(k) = \delta(k - k_0) \quad \text{loc.}$$



$$f(x) = \frac{e^{ik_0 x}}{\sqrt{2\pi}} \rightsquigarrow \text{deloc.}$$

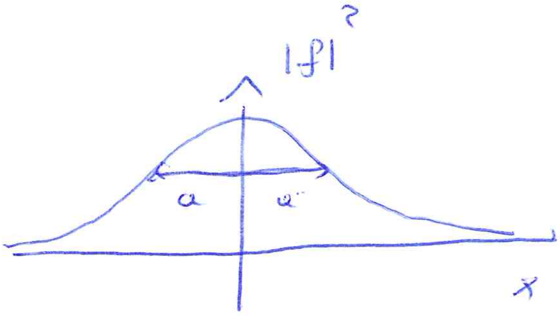
These are interesting mathematical limits, but $f(x)$ does not correspond to a w.f... namely, $|f(x)|^2$ is not normalised $\left(\int_{-\infty}^{\infty} \delta^2(x-x_0)\, dx = \delta(0) = \infty\right)$.

(A formal norm. procedure is possible, but it brings us too far... while, intuitively, $f(x) \sim \delta(x-x_0)$ means that the particle is 100% in x_0 and nowhere else!!!)

3. $\boxed{\text{Gaussian wave packet}}$ (We do it as a exercise)

$$\psi(0,x) \equiv f(x) = N e^{-x^2/a^2}$$

- Determine N



$$(f,f) = 1$$

$$|N|^2 \underbrace{\int_{-\infty}^{\infty} e^{-2x^2/a^2} dx}_{} = 1$$

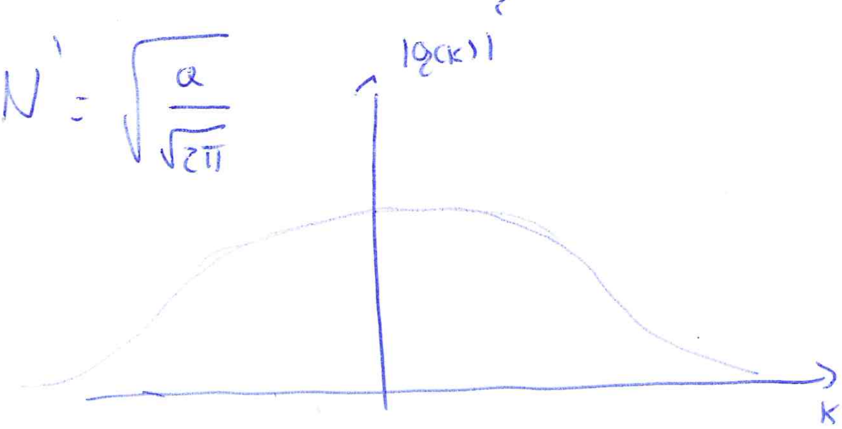
$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}}$$

$$|N|^2 \, a\sqrt{\frac{\pi}{2}} = 1 \rightarrow N = \sqrt{\frac{1}{a}\sqrt{\frac{2}{\pi}}}$$

- Determine $g(k)$

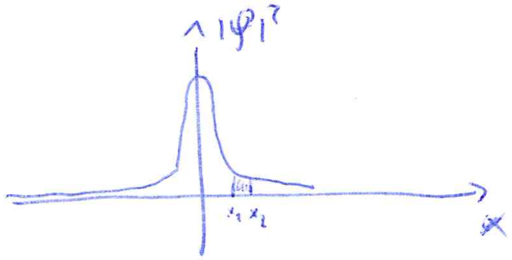
$$g(k) = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi}} e^{-ikx} dx = N' e^{-\frac{k^2a^2}{4}}$$

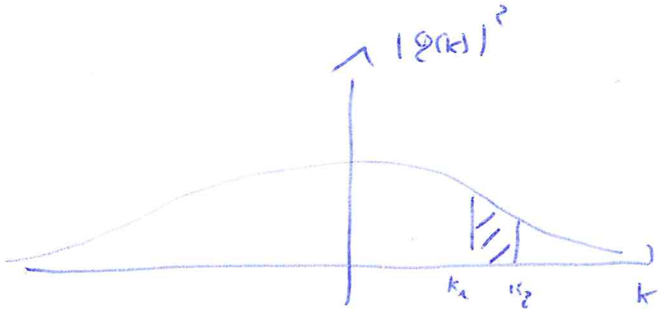
$$N' = \sqrt{\frac{a}{\sqrt{2\pi}}}$$



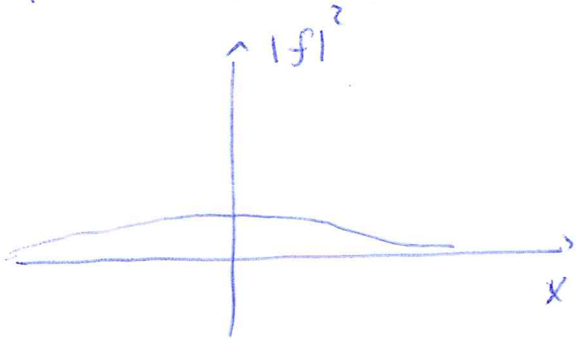
$$\boxed{(f,f) = (g,g) = 1}$$

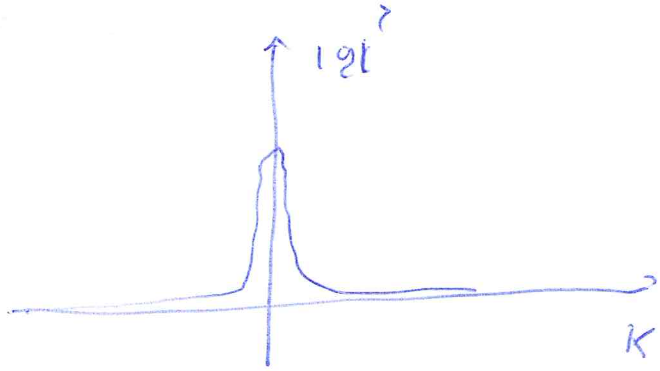
If a is small:





If a is large





$$\int_{x_1}^{x_2} |f|^2 dx$$ is the prob. to find the particle between (x_1, x_2) by a position of the meas.

Now, $|g|^2$ has a completely analog. interpretation:

$$\int_{K_1}^{K_2} |g(K)|^2 dK$$ is the prob. to find the momentum between $\hbar K_1$ and $\hbar K_2$!

The more we "know" about the position, the "less" we know about the momentum.

But... we either measure the momentum or the position...
Then, we can for instance in 50% of cases measure $\hat{x}$, in 50% of cases $\hat{p}$
and we realise that...

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, g(k) e^{-ikx}$$

$$H\varphi_i = E_i \varphi_i$$

$$\hat{P}_x\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right) = \hbar k\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right)$$

$$(f,f) = 1 = \sum_{i=1}^{\infty} |c_i|^2$$

$$(f,f) = 1 = \int_{-\infty}^{\infty} |g(k)|^2 dk$$

$$(f, Hf) = \sum_{i=1}^{\infty} |c_i|^2 E_i$$

$$(f, \hat{P}_x f) = \int_{-\infty}^{\infty} \hbar k\, |g(k)|^2 dk \quad \left(= \int_{-\infty}^{\infty} f^* \hat{P}_x f\right)$$

$$(\varphi_i, \varphi_j) = \delta_{ij}$$

$$\left(\frac{e^{ikx}}{\sqrt{2\pi}}, \frac{e^{ik'x}}{\sqrt{2\pi}}\right) = \delta(k-k')$$

$$(f, H^2 f) = \sum_{i=1}^{\infty} |c_i|^2 E_i^2$$

$$(f, \hat{P}_x^2 f) = \int_{-\infty}^{\infty} \hbar^2 k^2 |g(k)|^2 dk$$

The reason is that $\hat{P}_x$ has a cont. spectrum of eigenvalues.

$$\hat{P}_x\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right) = \hbar k\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right).$$

$\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right.$ is $/\,|\,|^2 = \frac{1}{2\pi} \rightarrow$ Not norm. to.

So, a Fourier transf. is nothing else than a recasting of $f(x)$ in the basis of eigenfunctions of the momentum.

In the present example:

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = (f, x^2 f) = \frac{a^2}{4}$$

$$\boxed{\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{2}}$$

$$\langle p \rangle = (f, \hat{p}_x f) = \int_{-\infty}^{\infty} f^*(x) \left(-i\hbar \frac{\partial}{\partial x} f(x)\right) dx =$$

$$= (\varphi(k), \hbar k \varphi(k)) = 0$$

$$\langle p^2 \rangle = (f, \hat{p}_x^2 f) = (\varphi, \hbar^2 k^2 \varphi(k)) = \frac{\hbar^2}{a^2}.$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{a}$$

$$\boxed{\Delta p \cdot \Delta x = \frac{\hbar}{2}}$$

In general $\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$

Note however:

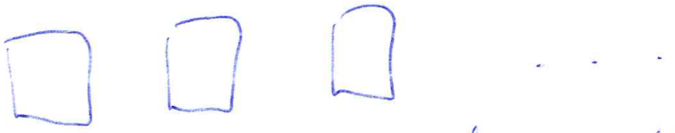
$\Delta x \Delta p = \frac{\hbar}{2}$

means that we have

Δx is not the exp. error...

$\Rightarrow$ $\delta x \ll \Delta x$ (otherwise we couldn't see anything...).

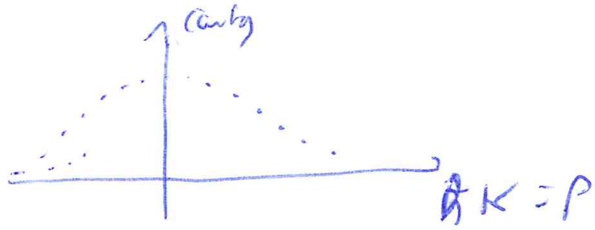
$\delta p \ll \Delta p$



100 boxes with the equally prep. $\psi(0,x)$

50 times we meas. $\hat{x}$

50 times we meas. $\hat{p}$



$\delta x \, \delta p < \hbar/2$ ...

$\boxed{\text{But } \Delta x \Delta p \geq \hbar/2}$

What about the eigenfunctions of x?

$$\hat{X}\, \varphi(x) = x\, \varphi(x) \equiv x_0\, \varphi(x)$$

↳ eigenf. with eigenvalue x_0.

$$\boxed{\varphi(x) = \delta(x - x_0)}$$

A generic $f(x)$ can be written as:

$$f(x) = \int_{-\infty}^{\infty} f(x_0)\, \delta(x - x_0)\, dx_0 \qquad (f, f) = 1$$

"sum over a basis of eigenfunctions of the position operator.

$\varphi(x) = \delta(x - x_0)$ is not normalised!

$$\int_{x_1}^{x_2} |f(x)|^2\, dx \longmapsto \text{prob. to see } f \text{ between } x_1 \text{ and } x_2.$$

FAZIT: x and p_x have a cont. spectrum of eigenvalues.