$\mathbb{C}^N$ and $L^2(a,b)$

$\mathbb{C}^N$: vector space (Just as $\mathbb{R}^N$ but with complex nrs).

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \in \mathbb{C}^N, \quad x_i \in \mathbb{C}$$

(nb: one can do it in a more abstract way, without a particular representation...)

(...but I prefer this "more practical" (although not general) approach...)

Scalar product:

$$\vec{x} \cdot \vec{y} = (\vec{x}, \vec{y}) = \sum_{i=1}^{N} x_i^* y_i = (x_1^*, x_2^*, \dots, x_N^*) \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

Lenght squared

$$\vec{x} \cdot \vec{x} = (\vec{x}, \vec{x}) = \sum_{i=1}^{N} x_i^* x_i = \sum_{i=1}^{N} |x_i|^2$$

Basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \vec{e}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \rightarrow \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

$$\vec{x} = \sum_{i=1}^{N} x_i \vec{e}_i$$

I could, of course, define a new ONC basis $\vec{v}_i$ / $\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$

$$\vec{x} = \sum_{i=1}^{N} x_i \vec{e}_i = \sum_{i=1}^{N} \tilde{x}_i \vec{v}_i \qquad \text{(change of basis)}$$

A philosophical question

$\mathbb{C}$: Why complex numbers in nature???

or better in QM?

In ED $\mapsto$ I may introduce complex nrs, but this is just a mathematical "trick"...

A_μ is "real"

$\vec{E}, \vec{B}$ are also real quantities.

How, the Schrödinger eq. naturally contains an i: $+i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$

ψ must be a complex quantity.

Is the need of $\mathbb{C}$ in QM necessary? Why is it naturally part of the QM?

Table

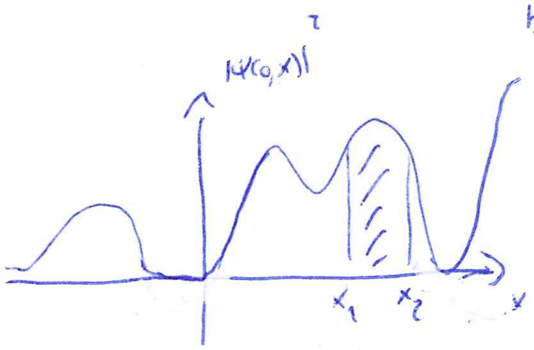
$\mathbb{C}^N$	$L^2(a,b)$
$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad x_i \in \mathbb{C}$	$f(x) : [a,b) \subseteq \mathbb{R} \mapsto \mathbb{C}$
$\lvert\vec{x}\rvert^2 = \vec{x}^2 = \sum_{i=1}^{N} \lvert x_i\rvert^2 =$ "length squared"	$\int_a^b \lvert f(x)\rvert^2 dx < \infty$
$\vec{x}\cdot\vec{y} = (\vec{x},\vec{y}) = \sum_{i=1}^{N} x_i^* y_i$ (scalar product)	$(f,g) = \int_a^b f^*(x)\, g(x)\, dx \quad ; \quad (f,f) = \int_a^b \lvert f\rvert^2 dx$
$\lvert\vec{x}\cdot\vec{y}\rvert^2 \leq \lvert\vec{x}\rvert^2 \lvert\vec{y}\rvert^2$ (Schwarz)	$\lvert(f,g)\rvert \leq (f,f)(g,g)$
$\begin{cases} \vec{e}_i & \text{"basis"} \\ \vec{e}_i\cdot\vec{e}_j = \delta_{ij} & \text{ON basis} \end{cases}$	$\begin{cases} \varphi_i & \text{basis} \\ (\varphi_i,\varphi_j) = \int_a^b \varphi_i^* \varphi_j\, dx = \delta_{ij} & \text{ON} \end{cases}$
$\vec{x} = \sum_{i=1}^{N} x_i \vec{e}_i$	$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i$
$x_i = (\vec{e}_i,\vec{x}) = \vec{e}_i\cdot\vec{x}$	$c_i = (\varphi_i, f)$
$\vec{x}^2 = \sum_{i=1}^{N} x_i^2$	$(f,f) = \int_a^b \lvert f\rvert^2 dx = \sum_{i=1}^{\infty} \lvert c_i\rvert^2$ (Parseval!)
$\vec{x}\cdot\vec{y} = (\vec{x},\vec{y}) = \sum_{i=1}^{N} x_i^* y_i$	$(f,g) = \sum_{i=1}^{\infty} c_i^* d_i \quad (g = \sum_{i=1}^{\infty} d_i \varphi_i)$
$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$	$f \doteq \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$ is just as $\mathbb{C}^{N\mapsto\infty}$
$\begin{cases} \{\vec{e}_i\},\ \{\vec{v}_i\} & \text{two basis:} \\ \vec{x} = \sum_{i=1}^{N} x_i \vec{e}_i = \sum_{i=1}^{N} \tilde{x}_i \vec{v}_i \end{cases}$	$\begin{cases} \{\varphi_i\},\ \{\eta_i\} \\ f = \sum_{i=1}^{\infty} c_i \varphi_i = \sum_{i=1}^{\infty} \tilde{c}_i \eta_i \end{cases}$
$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix},\ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \ldots \quad \vec{e}_i\cdot\vec{e}_j = \delta_{ij}$	$\varphi_i(x) = \left\{ \frac{e^{imx}}{\sqrt{2\pi}} \right\}$ for $L^2(-\pi,\pi)$; $(\varphi_i,\varphi_j) = \delta_{ij}$ ↳ extension number... later

Link to QM

$\psi(t,x)$ wave function.

$\psi(0,x) \in L^2(-\infty,\infty)$ and $\int_{-\infty}^{\infty} |\psi(0,x)|^2 dx = 1$

(not only finite (that is math) but 1 → physical requirement!)



$\int_{x_1}^{x_2} |\psi(0,x)|^2 dx$ is the prob. to find the particle between "x_1 and x_2" for a meas. of the position of the particle ⇒ electron.

$\hat{A} : L^2(-\infty,\infty) \mapsto L^2(-\infty,\infty)$ is an operator.

Some operators correspond to physical observables (as energy, momentum, angular momentum, position)

$\hat{A} f = \lambda f$ ↦ eigenvalue equation → $\hat{A} \varphi_i = \lambda_i \varphi_i$ $\{\varphi_i\}$ ON basis of L^2.

(Operators corresp. to observables are hermitian ↦ λ_i real).

$$\psi(0,x) = \sum_{i=1}^{\infty} c_i \varphi_i(x)$$ ⇒ this is surely possible because $\{\varphi_i\}$ is a basis.

$$(\psi(0,x), \psi(0,x)) = \int |\psi(0,x)|^2 dx = 1 \overset{\text{Parseval}}{=} \sum_{i=1}^{\infty} |c_i|^2$$

$\hat{A}\varphi_i = \lambda_i \varphi_i$

$$\psi(0^-,x) = \sum_{i=1}^{\infty} c_i \varphi_i$$

$$= c_1\varphi_1 + c_2\varphi_2 + \dots$$

$\boxed{\text{Meas. of } \hat{A} \text{ at } t=0}$ →

The result is λ_1 with prob. $|c_1|^2$; $\psi(0^+,x) = \varphi_1(x)$

" " λ_2 with prob. $|c_2|^2$; $\psi(0^+,x) = \varphi_2(x)$

. . . .

" " λ_n with prob. $|c_n|^2$; $\psi(0^+,x) = \varphi_n(x)$

. . .

Now, we may prepare the particle in the same initial state $\psi(0^-,x)$ and repeat the experiment many times. The average is

$$\boxed{\langle \hat{A} \rangle = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i}$$

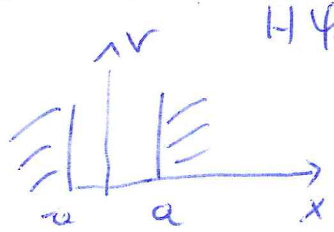
Namely, in $|c_1|^2$ of cases we find λ_1, $|c_2|^2$ of cases λ_2, ...

Example:

$$\psi(0,x) = \sqrt{\tfrac{1}{3}}\,\varphi_1(x) + c_2\,\varphi_2(x)$$

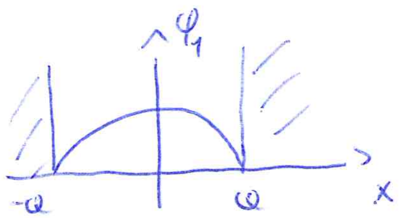
$\varphi_1(x)$ and $\varphi_2(x)$ eigenvectors of $\hat{H}$

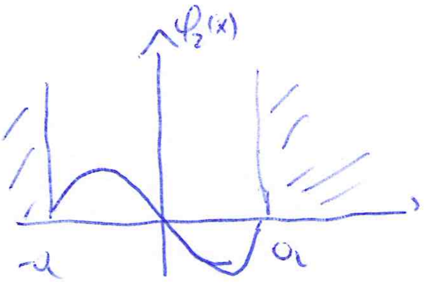
$$\hat{H}\varphi_1 = E_1\varphi_1$$
$$\hat{H}\varphi_2 = E_2\varphi_2$$



For instance, we can take

$$\begin{cases} \varphi_1(x) = \sqrt{\frac{1}{a}}\cos\frac{\pi}{2a}, \quad |x|\leq a, \quad 0 \text{ otherwise} \quad ; \quad E_1 = \dfrac{\hbar^2\pi^2}{8ma^2} \\ \varphi_2(x) = \sqrt{\frac{1}{a}}\sin\left(\frac{\pi}{a}\right), \quad |x|\leq a, \quad \text{" " } \quad ; \quad E_2 = \dfrac{\hbar^2\pi^2}{2ma^2} = 4E_1 \end{cases}$$





- Determine c_2

$$\int_{-\infty}^{\infty}|\psi(0,x)|^2dx = \int_{-a}^{a}|\psi(0,x)|^2dx = \sum_{i=1}^{\infty}|c_i|^2 = |c_1|^2 + |c_2|^2 = 1 = \frac{1}{3} + |c_2|^2$$

Parseval

$$|c_2|^2 = \frac{2}{3}$$

$$|c_2| = \sqrt{\frac{2}{3}} \longrightarrow c_2 = \sqrt{\frac{2}{3}}\, e^{i\phi}$$ ϕ "undetermined"

Ergo:

$$\psi(0^-, x) = \sqrt{\frac{1}{3}}\, \varphi_1(x) + \sqrt{\frac{2}{3}}\, e^{i\phi} \varphi_2(x)$$

At $t=0$ we measure the energy. The result is

with prob. $1/3$
we find E_1.
$t=0^+$, $\psi(0^+, x) = \varphi_1(x)$

with prob $\frac{2}{3}$
we find E_2
$\psi(0^+, x) = \varphi_2(x)$

$$\langle E \rangle = \langle \hat{H} \rangle = \sum_{i=1}^{2} |c_i|^2 E_i = \frac{1}{3} E_1 + \frac{2}{3} E_2$$

Operators in $\mathbb{C}^N$: BASIC DEFS

$\hat{A} : L^2(a,b) \mapsto L^2(a,b)$ (for instance: $\frac{d}{dx}$ or x $\begin{cases} \frac{d}{dx} : f(x) \mapsto f'(x) \\ x : f(x) \mapsto x f(x) \end{cases}$

But let us first go back to $\underline{\underline{\mathbb{C}^N}}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \qquad x_i \in \mathbb{C}$$

An operator is such that

$$\hat{A} : \mathbb{C}^N \mapsto \mathbb{C}^N \quad ; \quad \hat{A}\vec{x} = \vec{y} \in \mathbb{C}^N$$

We then realize that an operator is a $N \times N$ complex matrix

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{N1} & & & a_{NN} \end{pmatrix} \qquad a_{ij} \in \mathbb{C}$$

$$\hat{A}\vec{x} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1N}x_N \\ \vdots \\ a_{N1}x_1 + \ldots + a_{NN}x_N \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

Example:

$$\hat{A} = \begin{pmatrix} 1 & i \\ 0 & 7+2i \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 3+4i \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & i \\ 0 & 7+2i \end{pmatrix}}_{\hat{A}} \underbrace{\begin{pmatrix} 1 \\ 3+4i \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 1+3i-4 \\ (7+2i)(3+4i) \end{pmatrix}}_{\vec{y}}$$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \ldots \quad \vec{e}_J = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}; \ldots \quad \vec{e}_N = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\hat{A}\vec{e}_1 = \begin{pmatrix} a_{11} & & a_{1N} \\ a_{21} & & \\ a_{N1} & & a_{NN} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} = a_{11} \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_1} + a_{21} \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_2} + \ldots + a_{N1} \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}}_{\vec{e}_N}$$

$$\hat{A}\vec{e}_1 = \sum_{K=1}^{N} a_{K1} \vec{e}_K = a_{K1} \vec{e}_K$$ (sum understood)

Generalise $\boxed{\hat{A}\vec{e}_J = a_{KJ} \vec{e}_K}$

$\vec{z}, \vec{x} \in \mathbb{C}^N \Rightarrow$ one oftens calculate:

$$(\vec{z}, \hat{A}\vec{x}) = \vec{z}^* \cdot \hat{A}\vec{x}$$

Now, first we can evaluate

$$(\vec{e}_i, \hat{A}\vec{e}_j) = \left(\vec{e}_i, a_{kj}\vec{e}_k\right) = a_{kj}\delta_{ik} = a_{ij}$$

$$\boxed{\left(\vec{e}_i, \hat{A}\vec{e}_j\right) = a_{ij}} \mapsto \text{elements of the matrix}$$

But then:

$$\left(\vec{z}, \hat{A}\vec{x}\right) = \left(z_i\vec{e}_i, \hat{A}(x_j\vec{e}_j)\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} z_i^* x_j a_{ij}$$

$$= (z_1^*, \dots, z_N^*) \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

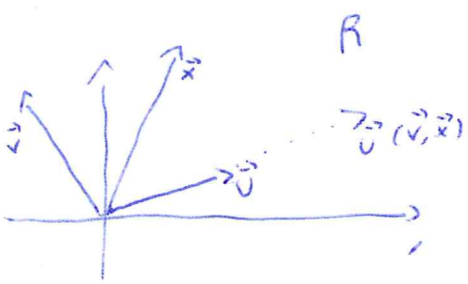
But... we have the scalar product...

Take $\vec{u}, \vec{v} \in \mathbb{C}^N$ and define:

$$\vec{u}(\vec{v}, \cdot) : \mathbb{C}^N \longmapsto \mathbb{C}^N$$

in the following way:

$$\vec{x} \longmapsto \vec{u}(\vec{v}, \vec{x})$$



This is a well defined operator. You may say: ok, but what for?

Each operator $\hat{A}$ can be expressed in that way!

$$\left\{ \begin{array}{l} \hat{A} = \sum_{i,j=1}^{N} a_{ij} \, \vec{e}_i (\vec{e}_j, \cdot) \\ \\ 1 = \sum_{i=1}^{N} \vec{e}_i (\vec{e}_i, \cdot) \end{array} \right.$$

Operators in $\mathbb{C}^N$; $t, *, \dagger$

$$\hat{A}: \mathbb{C}^N \mapsto \mathbb{C}^N \quad ; \quad \hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}$$

$$\hat{A}^t = \begin{pmatrix} a_{11} & a_{N1} \\ a_{1N} & a_{NN} \end{pmatrix} \quad ; \quad \hat{A}^{t^t} = \hat{A}$$

$$\hat{A}^* = \begin{pmatrix} a_{11}^* & a_{1N}^* \\ a_{N1}^* & a_{NN}^* \end{pmatrix} \quad ; \quad \left(\hat{A}^*\right)^* = \hat{A}$$

$$\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$$

$$\hat{A}^\dagger = \hat{A}^{t^*} = \begin{pmatrix} a_{11}^* & \cdots & a_{N1}^* \\ \vdots & \ddots & \vdots \\ a_{1N}^* & \cdots & a_{NN}^* \end{pmatrix} \rightarrow \boxed{(\vec{z}, \hat{A}\vec{x}) = (\hat{A}^\dagger \vec{z}, \vec{x})}$$

$$\hat{A}^{-1} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1N} \\ \vdots & & \vdots \\ \alpha_{N1} & \cdots & \alpha_{NN} \end{pmatrix} \quad / \quad \hat{A}^{-1}\hat{A} = \hat{A}\hat{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & & 1 \end{pmatrix} = 1_N$$

An operator $\hat{A}$ is hermitian if:

$$\hat{A} = \hat{A}^\dagger$$

An operator $\hat{U}$ is unitary if

$$\hat{U}^\dagger = \hat{U}^{-1}$$

A unitary operator $\hat{U} = e^{i\hat{A}}$ where $\hat{A}$ is unitary.

$$e^{i\hat{A}} = 1 + i\hat{A} + \frac{i^2}{2}\hat{A}^2 \cdots = \sum_{n=0}^{\infty} \frac{(i\hat{A})^n}{n!}$$

$$\hat{A}^{+} = A^{t\,*}$$

$$(\vec{z}, \hat{A}\vec{x}) = (z_1^*, \dots, z_N^*) \begin{pmatrix} a_{11} & a_{1N} \\ a_{N1} & a_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} =$$

$$= \left[\hat{A}^{+} \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \right]^{+} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = (\hat{A}^{+}\vec{z}, \vec{x})$$

$\underline{L^2(a,b)}$

$$\hat{A} : L^2(a,b) \longmapsto L^2(a,b).$$

$\{\varphi_i\}$ ON basis

The elements of $\hat{A}$ are given by

$$(\varphi_i, \hat{A}\varphi_j) = \int_a^b dx\, \varphi_i^*(\hat{A}\varphi_j(x))$$

$$\hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1N} & \cdots \\ \vdots & \ddots & & \\ a_{N1} & & a_{NN} & \\ \vdots & & & \ddots \end{pmatrix}$$

$\boxed{\text{Again } N \longmapsto \infty}$

Definition of $\underline{\hat{A}^+}$:

$$(\varphi_i, \hat{A}\varphi_j) = (\hat{A}^+\varphi_i, \varphi_j) \qquad \forall i,j$$

$$\int_a^b \varphi_i^*(\hat{A}\varphi_j)\,dx = \int_a^b (\hat{A}^+\varphi_i)^*\varphi_j\,dx$$

An hermitian operator $\hat{A}^+ = \hat{A}$ is then such that

$$(\varphi_i, \hat{A}\varphi_j) = (\hat{A}\varphi_i, \varphi_j)$$

$$\int_a^b \varphi_i^*\hat{A}\varphi_j\,dx = \int_a^b (\hat{A}\varphi_i)^*\varphi_j\,dx$$

In term of the matrix

$\hat{A}^{t*} = \hat{A}$ (valid also for $N\to\infty$)

Example:

operator $\hat{x}$:

$$\hat{x} : f(x) \longrightarrow x f(x)$$

$$\left(\hat{x} : x^2 \longmapsto x^3 \right)$$

Is this hermitian?

$$(\varphi_i, x\varphi_j) = \int_a^b \varphi_i^* x \varphi_j \, dx = \int_a^b (x\varphi_i)^* \varphi_j \, dx = (x\varphi_i, \varphi_j)$$

$$\hat{x}^+ = \hat{x}$$

$$\hat{\frac{d}{dx}} : f(x) \longmapsto \frac{df}{dx}$$

$$\int_a^b \varphi_i^* \left(\frac{d}{dx} \varphi_j(x) \right) dx = \left[\varphi_i^* \varphi_j \right]_a^b - \int_a^b \frac{d\varphi_j^*}{dx}$$

$\frac{d}{dx}$ Not herm...

If we require that $\varphi_i(x)$ are periodic: $\varphi_i(a) = \varphi_i(b)$, we get:

$$\left(\varphi_i, \frac{d\varphi_j}{dx} \right) = \left(-\frac{d\varphi_i}{dx}, \varphi_j \right)$$

$$\left(\frac{d}{dx} \right)^+ = -\frac{d}{dx} \qquad \text{Antihermitian.}$$

But then... if we consider the operator

$$\hat{P}_x = -i\hbar \frac{\partial}{\partial x}$$

Then, this is hermitian:

$$(\varphi_i, \hat{P}_x \varphi_j) = \int_a^b \varphi_i^*(x) \left(-i\hbar \frac{\partial \varphi_j}{\partial x}\right) dx = \underbrace{\left[-i\hbar \varphi_i^*(x) \varphi_j(x)\right]_a^b}_{=0}$$

$$+ i\hbar \int_a^b \frac{\partial \varphi_i^*}{\partial x} \varphi_j \, dx =$$

$$= \int_a^b \left(-i\hbar \frac{\partial \varphi_i}{\partial x}\right)^* \varphi_j \, dx$$

$$(\varphi_i, \hat{P}_x \varphi_j) = (\hat{P}_x \varphi_i, \varphi_j) \quad \forall i, j$$

$$(f, \hat{P}_x g) = (\hat{P}_x f, g) \qquad \forall f, g.$$

$\hat{A} = A(x, P_x)$ -> generic operator $L^2(a,b) \mapsto L^2(a,b)$

$$A = \sum_{n,m} c_{nm} \hat{x}_i^n \hat{P}_j^m .$$

IMPORTANT THEOREM:

$$\boxed{\hat{A} \text{ hermitian} \longmapsto \text{eigenvalues are real}}$$

We search for a particular set $\{\varphi_i\}$ /

$$\hat{A}\varphi_i = \lambda_i \varphi_i \qquad i = 1, 2, \dots$$

$$(\varphi_i, \varphi_i) = 1$$

$$\boxed{(\varphi_i, \hat{A}\varphi_i) = \lambda_i} \qquad \left(= \int_a^b \varphi_i^* \varphi_i \lambda_i \, dx = \lambda_i \int_a^b |\varphi_i|^2 dx = 1 \right)$$

But, if $\hat{A}$ is hermitian:

$$(\varphi_i, \hat{A}\varphi_i) = (\hat{A}\varphi_i, \varphi_i) = (\lambda\varphi_i, \varphi_i) = \int_a^b (\lambda_i \varphi_i)^* \varphi_i \, dx = \lambda_i^* \int_a^b |\varphi_i|^2 dx =$$
$$= \lambda_i^*$$

Ergo: $\boxed{\lambda_i^* = \lambda_i}$

This mathematical property is very important from a physical point of view. It assures that the quantities I measure are "real": λ_i are the results of my measurement (as for instance E_i).

2° IMPOR. THEOREM. $\hat{A}$ herm $\mapsto$ Eigenfunctions are $\perp$

$\hat{A}\varphi_i = \lambda_i \varphi_i$, $\hat{A}$ herm.

$$\begin{cases} \varphi_1 \text{ with eig. } \lambda_1 \\ \varphi_2 \text{ with eig. } \lambda_2 \neq \lambda_1 . \end{cases}$$

$\Rightarrow (\varphi_1, \varphi_2) = 0$

(authomatically $\perp$).

$$(\varphi_1, A\varphi_2) = (\varphi_1, \lambda_2 \varphi_2) = \lambda_2 (\varphi_1, \varphi_2)$$

BUT

$$(\varphi_1, A\varphi_2) = (A\varphi_1, \varphi_2) = (\lambda_1 \varphi_1, \varphi_2) = \lambda_1^* (\varphi_1, \varphi_2) = \lambda_1 (\varphi_1, \varphi_2)$$

Then:

$$\lambda_1 (\varphi_1, \varphi_2) = \lambda_2 (\varphi_1, \varphi_2) \Rightarrow \boxed{(\varphi_1, \varphi_2) = 0}$$

$$(\varphi_i, \varphi_j) = 0 \quad \forall\, \lambda_i \neq \lambda_j$$

$\boxed{\text{IMPORTANT PROPERTY}}$

$$\psi(0,x) = \sum_{i=1}^{\infty} c_i \varphi_i(x)$$

$$A\varphi_i(x) = \lambda_i \varphi_i(x) \qquad (\varphi_i, \varphi_j) = \delta_{ij}$$

$$(\psi(0,x), A\psi(0,x)) = \left(\sum_{i=1}^{\infty} c_i \varphi_i, A\left(\sum_j c_j \varphi_j\right)\right) =$$

$$= \left(\sum_i c_i \varphi_i, \sum_j c_j \varphi_j \lambda_j\right) = \sum_{i,j} c_i^* c_j \lambda_j \underbrace{(\varphi_i, \varphi_j)}_{\delta_{ij}}$$

$$= \sum_{i=1}^{\infty} |c_i|^2 \lambda = \langle \hat{A} \rangle_\psi$$

This is the average of the observable A if

$|c_1|^2$ of times: λ_1

$|c_2|^2$ " " : λ_2

. . . .

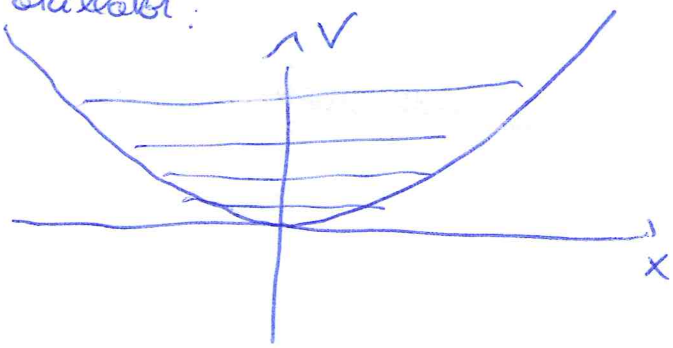
. . . . .

if I repeat the exp. many times (always starting from the same wf $\psi(0,x)$, the average of the result is $(\psi(0,x), A\psi(0,x))$

FINAL CONSIDERATIONS

Now, this is valid for the energy but also for the position and momentum.

Indeed, for what concerns the energy a confining potential such as the harmonic oscillator:



$$V = \frac{1}{2} m \omega^2 x^2 \rightarrow E_n = (n + \frac{1}{2}) \hbar \omega \qquad n = 0, 1, 2,$$

$$\varphi_n(x) \quad / \quad (\varphi_n, \varphi_m) = \delta_{nm}$$

This is true for each "confining potential"... the spectrum of H is <u>discrete</u>!

But what about the operator $\hat{x}$?

We realize that the eigenvalues are continuous... each x is ok.

It is a continuum ∞. Here, $(\psi(0,x), x\psi(0,x)) = \langle x \rangle_\psi =$

$$= \int_{-\infty}^{\infty} x \, |\psi(0,x)|^2 dx$$

Similarly,

Similarly, $P_x = -i\hbar \frac{\partial}{\partial x}$ has a cont. spec of eigenvalues (basically each real numbers, just as x).

However, the average is still defined:

$$(\psi(0,x), \hat{P}_x \psi(0,x)) = \int_{-\infty}^{\infty} \psi^*(0,x) \left(-i\hbar \frac{\partial \psi}{\partial x} \right) dx =$$

$$= \langle \hat{P}_x \rangle_{\psi(0,x)}$$

Next week:

$$\begin{cases} L^2(-\pi, \pi) \to \text{Fourier transform} \\ L^2(-\infty, \infty) \\ f(x) \Leftrightarrow A(k) \end{cases}$$