$L^2(a,b)$ SPACES

Let us consider functions of the type

$$f(x) : [a,b] \subseteq \mathbb{R} \mapsto \mathbb{C}$$

$$f(x) = U(x) + i\,V(x) \qquad \begin{cases} U(x) : [a,b] \subseteq \mathbb{R} \mapsto \mathbb{R} \\ V(x) : [a,b] \subseteq \mathbb{R} \mapsto \mathbb{R} \end{cases}$$

in QM:

$$\psi(t,x) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$$

$$\psi(0,x) : \mathbb{R} \mapsto \mathbb{C}$$

and often: $\psi(t,x) = f(t)\,\varphi(x)$ where: $\begin{matrix} f(t) : \mathbb{R} \mapsto \mathbb{C} \\ \varphi(x) : \mathbb{R} \mapsto \mathbb{C} \end{matrix}$

Space $L^2(a,b)$: $f(x)$ belongs to $L^2_{(a,b)}$ if the integral

$$\int_a^b |f(x)|^2\,dx = \int_a^b \left[|U(x)|^2 + |V(x)|^2\right] dx$$ is finite and well defined.

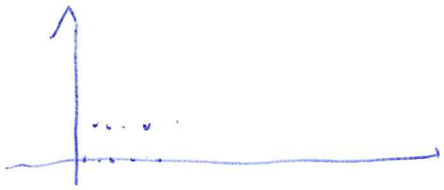
Achtung:

$\int \mapsto$ "Riemann integral"

one can also extend the concept of integral $\mapsto$ Lebesgue integral

$a = 0, b = 1$

$$d(x) = \begin{cases} 1 & x \in \mathbb{Q} \quad (x = n/m) \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$



this is "much larger" than $\mathbb{Q}$ ....

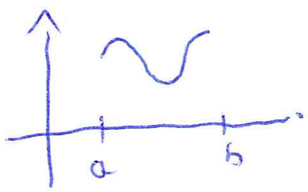
, uncountable vs countable

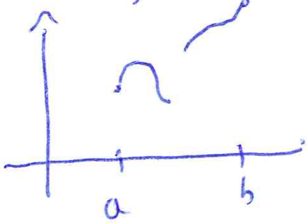
$$\int_0^1 d(x)\,dx = 0 \quad \text{(Lebesgue)}.$$

The correct definitions of these concepts go beyond our goals, but it is good to know that they exist.

Note:

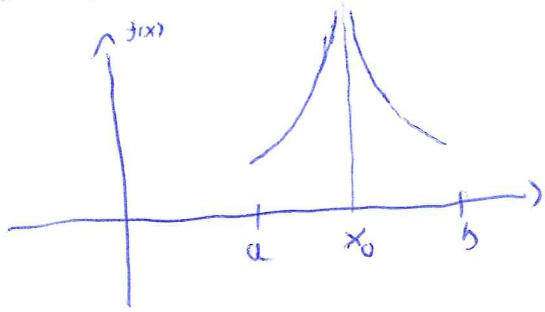
if $f(x)$ is not singular in any point of $[a,b]$ ↦ the integral exists.





But it can also diverges .... as long as the integral $\int |f(x)|^2 dx$ is finite, that is OK.

For instance: $x_0 \in (a,b)$



$$f(x) = \frac{1}{(x-x_0)^\alpha} \qquad |f(x)|^2 = \frac{1}{(x-x_0)^{2\alpha}}$$

$$\int_a^{x_0} \frac{1}{(x-x_0)^{2\alpha}} dx = \int_a^{x_0} (x-x_0)^{-2\alpha} dx = \left[-\frac{1}{2\alpha} (x-x_0)^{-2\alpha+1} \right]_a^{x_0}$$

$$1 - 2\alpha > 0 \mapsto 2\alpha < 1$$

$$\boxed{\alpha < 1/2}$$

$f(x) = \frac{1}{(x-x_0)^{1/4}}$ is OK.

also $f(x) = \frac{1}{(x-a)^{1/4}}$ is such...

Scalar product

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_N y_N$$

Let us now extend this idea to the functions belonging to $L^2(a,b)$.

$f(x), g(x) \in L^2(a,b)$

$$\boxed{(f(x), g(x)) = \int_a^b f^*(x)\, g(x)\, dx}$$

def of scalar product.

$(\cdot, \cdot): L^2 \times L^2 \mapsto \mathbb{C}$

PROPERTIES which follow from the def.

1) $(f, g)^* = (g, f)$

2) $(f, f) = \int_a^b f^*(x) f(x)\, dx = \int_a^b |f(x)|^2\, dx \geq 0$ and $= 0$ for $f(x) = 0 \ \forall x \in (a,b)$

"Lenght^2" of the vector $f(x)$

Just as $\vec{x}^2 = x_1^2 + x_2^2 + \dots + x_N^2$

3) $(f, \alpha g) = \alpha (f, g)$

$(\beta f, g) = \beta^* (f, g)$

4) $(f + h, g) = (f, g) + (h, g)$

$(f, g + h) = (f, g) + (f, h)$

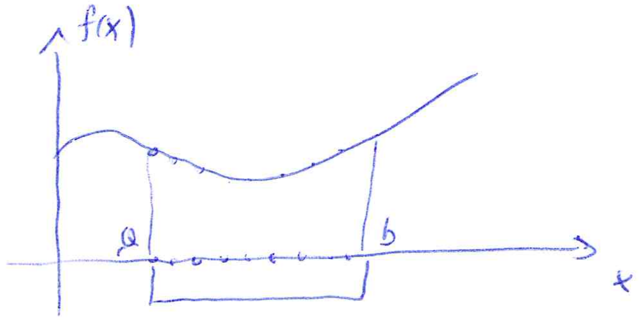
5) $|(f, g)|^2 \leq (f, f)(g, g)$ (Schwarz)

($\vec{x} \cdot \vec{y} = |\vec{x}| \cdot |\vec{y}| \cos \to |\vec{x} \cdot \vec{y}| = |\vec{x}| \cdot |\vec{y}|\, |\cos| \mapsto |\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$

$\hookrightarrow |\vec{x} \cdot \vec{y}|^2 \leq |\vec{x}|^2 \cdot |\vec{y}|^2$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$|\vec{x}|^2$$



$$\vec{v}(x) = \begin{pmatrix} f(a) \\ f(a+\delta x) \\ f(a+2\delta x) \\ \vdots \\ f(b-\delta x) \\ f(b) \end{pmatrix}$$

f(x) is like a vector with many entries... for a finite δx there are

$N = (b-a)/\delta x$ entries

For $\delta x \to 0$ there is a countless number of entries.

$|\vec{v}|^2 = \int |f(x)|^2 \, dx$ is the length of this "vector".

(Function $\to$ vector in this $L^2(a,b)$ space, in which a scalar product is perfectly well defined!)

Schwarz → PROOF.

$$\left| \int_a^b dx\, f^*(x)\, g(x) \right|^2 \leq \int_a^b dx\, |f(x)|^2 \int_a^b dx\, |g(x)|^2$$

Let us define

$$h(x) = f(x) + \lambda (g,f)\, g(x) \qquad \boxed{\lambda \in \mathbb{R}}$$

$$0 \leq (h,h) = \left(f + \lambda (g,f) g \,,\, f + \lambda (g,f) g \right) =$$

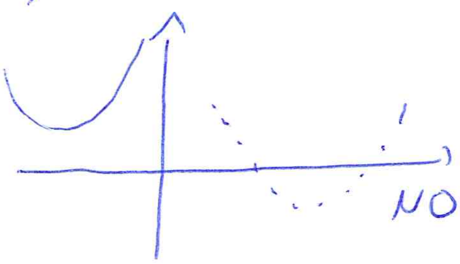
$$= (f,f) + \lambda (g,f)^* (g,f) + \lambda (g,f)(f,g)$$

$$+ \lambda^2 (g,f)^* (g,f)(g,g)$$

$$= (f,f) + 2\lambda |(f,g)|^2 + \lambda^2 (g,g) |(f,g)|^2 \geq 0 \quad (\forall \lambda)$$

We have something of the type

$$a\lambda^2 + 2b\lambda + c \geq 0 \qquad \forall \lambda$$



$$\lambda_{1,2} = -b \pm \sqrt{b^2 - ac}$$

$$b^2 - ac \leq 0$$

$$|(f,g)|^4 - (f,f)(g,g)\,|(f,g)|^2 \leq 0 \rightarrow |(f,g)|^2 \leq (f,f)(g,g)$$

q.e.d.

Two mathematical extensions:

1) $L^p(a,b)$

$f(x) \in L^p(a,b)$ if $\int_a^b |f(x)|^p dx$ exists. Interesting for mathematics, but for physics $p=2$ is the most important case.

$$\int_a^b |\psi(t,x)|^2 dx = 1 \quad \forall t$$

$$\int_a^b |f(x)|^2 dx \text{ exists} \mapsto \int_a^b |f(x)| dx \text{ exists as well}$$

2) one can introduce a weight function

$$L^2((a,b), \rho(x) \geq 0)$$

and define: $f(x)$ belongs to this space if

$$\int_a^b |f(x)|^2 \rho(x) dx$$

exists.

The scalar product can be extended by

$$(f,g) = \int_a^b f^*(x) g(x) \rho(x) dx$$

and all goes as before.

Let us consider a (countable) set of functions

$S = \{ \varphi_i(x) \}$ $\quad i = 1, 2, \dots \infty$ $\quad$ such that $\varphi_i(x) \in L^2(a,b)$.

- The set S is orthogonal if

$$(\varphi_i(x), \varphi_j(x)) = 0 \text{ for } i \neq j$$

- The set S is orthonormal if

$$(\varphi_i(x), \varphi_j(x)) = \delta_{ij}$$

- The set S is complete if each function $f(x) \in L^2(a,b)$

can be expressed as

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x)$$

where c_i are (complex) constant coefficients.

The latter means that:

$$\lim_{n \to \infty} \int_a^b \left| f(x) - \sum_{i=1}^{n} c_i \varphi_i(x) \right|^2 dx = 0$$

$S : \{\varphi_i(x)\}$ ONC

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x)$$

① Determination of c_i :

$$(\varphi_j, f) = \left(\varphi_j, \sum_{i=1}^{\infty} c_i \varphi_i\right) = \sum_{i=1}^{\infty} c_i \underbrace{(\varphi_j, \varphi_i)}^{\delta_{ij}} = \sum_{i=1}^{\infty} c_i \delta_{ij} = c_j$$

$$\boxed{c_j = (\varphi_j, f) = \int_a^b \varphi_j^*(x) f(x)\, dx}$$

② Completeness relation

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x) = \sum_{i=1}^{\infty} \int_a^b \varphi_i^*(x') f(x') dx' \cdot \varphi_i(x)$$

$$= \int_a^b \left(\sum_{i=1}^{\infty} \varphi_i^*(x') \varphi_i(x') \right) f(x') dx' = f(x)$$

$$\boxed{\sum_{i=1}^{\infty} \varphi_i^*(x') \varphi_i(x) = \delta(x'-x)}$$

This is also a completeness ..

Note:

Suppose that $S = \{ \varphi_i(x) \}$ is ON but not complete.

$$\sum_{i=1}^{N} \varphi_i^*(x') \varphi_i(x) = \delta(x-x') + \underbrace{r(x)}_{\text{rest function}}$$

If $f(x)$ is such that $f(x) = \sum_{i=1}^{N} c_i \varphi_i(x)$

Then, in this subset I effectively have $\sum_{i=1}^{\infty} \varphi_i^*(x') \varphi_i(x) \equiv \delta(x-x')$ ..

namely, $\int r(x) f(x) dx = 0$.

- Parseval equation:

$$(f,f) = \int_a^b f^*(x)\, f(x)\, dx$$

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x) \qquad c_i = (\varphi_i, f)$$

$$f^*(x) = \sum_{j=1}^{\infty} c_j^* \varphi_j^*(x)$$

$$(f,f) = \int_a^b \sum_{j=1}^{\infty} c_j^* \varphi_j^*(x) \sum_{i=1}^{\infty} c_i \varphi_i(x)\, dx =$$

$$= \sum_{j=1}^{\infty} c_j^* \sum_{i=1}^{\infty} c_i \underbrace{\int_a^b \varphi_j^*(x)\, \varphi_i(x)\, dx}_{\delta_{ij}} = \sum_{i=1}^{\infty} |c_i|^2 \quad (< \infty)$$

$L^2 \to$ finite integral

- Scalar product

$$(f,g) = \int_a^b f^*(x)\, g(x)\, dx$$

$$f = \sum_{i=1}^{\infty} c_i \varphi_i$$

$$g = \sum_{j=1}^{\infty} d_j \varphi_j$$

$$(f,g) = \sum_{i=1}^{\infty} c_i^* d_i$$

If you remind that a vector

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

we realise that a function $f(\vec{x}) \in L^2(a,b)$ is actually defined also by a vector of the type

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \end{pmatrix} \rightarrow \text{"}\infty\text{"} \ldots$$

with $\sum_{i=1}^{\infty} |c_i|^2 < \infty$.

As we shall see, this is nothing else than an Hilbert space!

$$f \leftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} \qquad g \leftrightarrow \begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix}$$

$$(f, g) = \sum_{i=1}^{\infty} c_i^* d_i$$

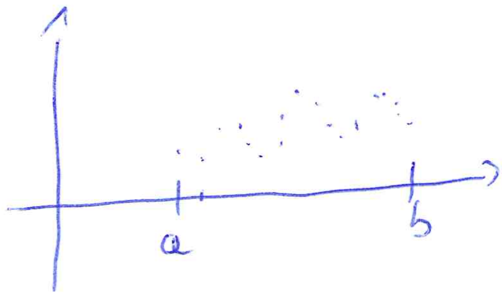
Just as:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{N} x_i y_i$$

Difference $\mapsto$ ∞ dimensionality (but still finite length)

complex coefficients.

Remind that



$$f(x) \longleftrightarrow \begin{pmatrix} f(a) \\ f(a+\delta x) \\ f(a+2\delta x) \\ \vdots \\ f(b-\delta x) \\ f(b) \end{pmatrix}$$

For $\delta x \mapsto 0$ this is a ∞ countless series of complex numbers ...

$$f(x) \longleftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

This is a ∞ but COUNTABLE series of complex nrs!!!

$$\left(\sum_{i=1}^{\infty} |c_i|^2 < \infty \right)$$

This is much easier!!!!

In QM

$$(f,f) = \int_a^b |f|^2 dx = 1 = \sum_{i=1}^{\infty} |c_i|^2$$

Closed system:

A system of function $S = \{ \varphi_i(x) \}$ is closed if

$$(f, \varphi_i) = 0 \quad \forall i \longmapsto f(x) = 0$$

closed system $\longleftrightarrow$ complete system

Corollary:

if $S = \{ \varphi_i \}$ is ON and closed $\longmapsto$ it is then ONC

$$\boxed{\sum_{i=1}^{\infty} \varphi_i^*(x') \varphi_i(x) = \delta(x-x')}$$

Completeness: more difficult to prove than closure...

EXAMPLE:

A complete system (but not ONS) in each finite $[a,b]$ is

$$\{x^n\} \quad n = 0, 1, 2, \dots$$

Taylor expansion.

For instance, let us take $[a,b] = [-1,1]$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

This is true for each $f(x)$ which is integrable.

Note,

$$(x,x) = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \neq 1$$

$$(1,x^2) = \int_{-1}^{1} x^2 dx = \frac{2}{3} \neq 0$$

$$(x,x^3) = \int_{-1}^{1} x^4 dx = \left(\frac{x^5}{5}\right)_{-1}^{1} = \frac{2}{5} \neq 0$$

obviously <u>NOT</u> complete !!!!

Fourier

In the interval $(-\pi, \pi)$ a ONC system is given by

$$\left\{ \frac{e^{imx}}{\sqrt{2\pi}} \right\} \quad m = 0, \pm 1, \pm 2, \pm 3$$

Recall:

$$\frac{e^{imx}}{\sqrt{2\pi}} = \frac{\cos(mx)}{\sqrt{2\pi}} + i \frac{\sin(mx)}{\sqrt{2\pi}}$$

$$(\varphi_m, \varphi_m) = \int_{-\pi}^{\pi} \frac{e^{-imx}}{\sqrt{2\pi}} \frac{e^{imx}}{\sqrt{2\pi}} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1$$

$n \neq m$

$$(\varphi_n, \varphi_m) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \frac{e^{imx}}{\sqrt{2\pi}} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx =$$

$$= \frac{1}{2\pi} \frac{1}{i(m-n)} \left(e^{i(m-n)\pi} - e^{-i(m-n)\pi} \right) = 0$$

Closed:

$$\int_{-\pi}^{\pi} f^*(x) \frac{e^{imx}}{\sqrt{2\pi}} dx = 0 \quad \forall m \quad \text{only if } f(x) = 0.$$

$$\boxed{\sum_{m=-\infty}^{\infty} \frac{e^{-imx'}}{\sqrt{2\pi}} \frac{e^{imx}}{\sqrt{2\pi}} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(x-x')} = \delta(x-x')}$$

Then, each $f(x)$ can be expressed as

$$f(x) = \sum_{m=-\infty}^{\infty} c_m \frac{e^{imx}}{\sqrt{2\pi}}$$

Note, $\frac{e^{imx}}{\sqrt{2\pi}}$ is an eigenstate of the "momentum operator"

$$\hat{P}_x = -i\hbar \frac{\partial}{\partial x}$$

with eigenvalue $m\hbar$.

We express then a generic wave function as a "sum of eigenfunctions with $\neq$ momentum".