Differential equations

obviously, diff. eq. in more dimensions can be very complicated.

They are expressed via "partial derivatives" ...

$$F\left(\frac{\partial^n f}{\partial x^n}, \frac{\partial^n f}{\partial x^{n-1}\partial y}, \dots \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0$$

"Field theory" and "Quantum field theory" deal with such equations.

Here we can simply discuss some basic examples.

Very Simple differential eq. of 1° order with partial derivatives

$$\frac{\partial f(x,y)}{\partial x} = y$$

Solution:

$$f(x,y) = yx + \underbrace{g(y)}_{\text{generic function...}}$$

There is not only an initial condition, but an initial function...

For instance

$$\boxed{f(1,y) = y^2}$$ "initial condition"

$$\Rightarrow g(y) = y^2 - y$$

$$\boxed{f(x,y) = xy + y^2 - y}$$

Let us now consider a system of 2 diff. eqs:

$$\begin{cases} \partial_x f = x^2 \\ \partial_y f = y^3 \end{cases}$$

$$f = \frac{x^3}{3} + g(y)$$

$$\partial_y f = \frac{\partial g}{\partial y} = y^3 \quad \rightarrow g(y) = \frac{y^4}{4} + c$$

$$\boxed{f(x,y) = \frac{x^3}{3} + \frac{y^4}{4} + c}$$

In this case one initial condition

(such as $f(0,0) = 0 \rightarrow c = 0$) is enough).

NOTE: Indeed, not always there are solutions out of such a system:

$$\begin{cases} \partial_x f = xy \\ \partial_y f = x \end{cases}$$

out of the first eq:

$$f(x,y) = \frac{x^2}{2} y + g(y)$$

$$\partial_y f = \frac{x^2}{2} + g'(y) = x$$

However, $g'(y)$ is a function of y and not of x...

If you try to force it:

$$g'(y) = x - \frac{x^2}{2}$$

$$g(y) = \left(x - \frac{x^2}{2}\right) y$$

You then spoil the 1° eq: $f(x,y) = \frac{x^2}{2} y + \left(x - \frac{x^2}{2}\right) y$

$$\boxed{\partial_x f = xy + (1-x) y}$$

Next step: 3 important equations with partial derivatives up to 2° order.

Laplace equation, wave equation, and diffusion equation.

Note that:

- 2° order are "natural"... they occur in most physical problems.

(Newton eq. is of 2° order, but also the eq. for electric and magnetic fields are such... this indeed goes back to the fact that the $\mathcal{L}$ (Lagrangian) contains only terms with 1° order).

- We study the "simple form of the sols." and also we do not take into account "sources"...

Operator: $\Delta = \partial_x^2 + \partial_y^2$

Let us study

$$\boxed{\Delta f(x,y) = 0}$$

A simple solution can be found by including only terms up to 1° order and a mix-term:

$$f(x,y) = \alpha + \beta x + \gamma y + \delta x y$$

In fact.

$$\begin{cases} \partial_x^2 f(x,y) = 0 \\ \partial_y^2 f(x,y) = 0 \end{cases}$$

separately!

However this is only a "limited" form of the possible solution.

In fact, we can have $f(x,y)$ such that

$$\partial_x^2 f = -\partial_y^2 f \neq 0$$

$$\Downarrow$$

$$(\partial_x^2 + \partial_y^2) f = 0$$

A trick, often used to solve partial diff. eq., is to make a "separationansatz":

$$\boxed{f(x,y) = U(x) W(y)}$$

Then:

$$0 = (\partial_x^2 + \partial_y^2) f(x,y) = (\partial_x^2 + \partial_y^2) U(x) W(y)$$

$$= \left[\partial_x^2 U(x)\right] W(y) + U(x) \left[\partial_y^2 W(y)\right] = 0$$

$$\left[\partial_x^2 U(x)\right] W(y) = - U(x) \left[\partial_y^2 W(y)\right]$$

$$\frac{\partial_x^2 U(x)}{U(x)} = - \frac{\partial_y^2 W(y)}{W(y)}$$

There is only one possibility!

$$\frac{\partial_x^2 U(x)}{U(x)} = - \frac{\partial_y^2 W(y)}{W(y)} = K = \text{const.}$$

For simplicity, let us consider $K > 0$!

$$\frac{\partial_x^2 U(x)}{U(x)} = K$$

$$\boxed{\frac{d_x^2 U}{dx^2} - K U(x) = 0}$$

$$U(x) = \left(\alpha e^{\sqrt{K}x} + \beta e^{-\sqrt{K}x}\right).$$

For the other eq. we get:

$$\frac{d^2 W}{dy^2} + K W(y) = 0$$

$$W(y) = \gamma \sin(\sqrt{K}\, y) + \delta \cos(\sqrt{K}\, y)$$

Ergo the full solution (with separation):

$$\boxed{f(x,y) = \left(\alpha e^{\sqrt{K}x} + \beta e^{-\sqrt{K}x}\right)\left(\gamma \sin(\sqrt{K}\, y) + \delta \cos(\sqrt{K}\, y)\right)}$$

Diffusion equation :

$$\boxed{\left[\kappa\partial_t - \partial_x^2\right] \phi(t,x) = 0}$$

Write down the "separation postulate"

$$\phi(t,x) = u(x)\, w(t)$$

$$\Downarrow$$

Find a solution of this form.

Exp. Ansatz

$$\phi(t,x) = e^{\alpha t + \beta x}$$

$$\partial_t \phi(t,x) = \alpha e^{\alpha t + \beta x}$$

$$\begin{cases} \partial_x \phi(t,x) = \beta e^{\alpha t + \beta x} \\ \partial_x^2 \phi(t,x) = \beta^2 e^{\alpha t + \beta x} \end{cases}$$

Plug it in:

$$\partial_t \phi = \alpha e^{\alpha t + \beta x} = \partial_x^2 \phi = \beta^2 e^{\alpha t + \beta x}$$

$$\Rightarrow \boxed{\alpha = \beta^2}$$

$$\phi(t,x) = N e^{\beta^2 t + \beta x}$$

$$\beta = iK \quad \alpha = -K^2$$

$$\phi(t,x) = N e^{-K^2 t + iKx}$$

$$\Downarrow$$

$$\phi(t,x) = \int_{-\infty}^{\infty} \frac{dK}{2\pi} f(K) e^{-K^2 t + iKx}$$

Schrödinger eq:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi$$

$$\psi(t,x) = e^{\alpha t + \beta x}$$

$$i\partial_t\psi = i\alpha\, e^{\alpha t + \beta x}$$

$$\partial_x\psi = \beta\, e^{\alpha t + \beta x}$$

$$\partial_x^2\psi = \beta^2 e^{\alpha t + \beta x}$$

$$i\alpha\, e^{\alpha t + \beta x} = -\frac{1}{2m}\beta^2 e^{\alpha t + \beta x}$$

$$\alpha = \frac{+i}{2m}\beta^2 \qquad \beta = +ik$$

$$\alpha = \frac{+i}{2m}(-k^2) = -i\frac{k^2}{2m}$$

$$\boxed{\psi(t,x) = e^{-i[\omega(k)t - kx]}}$$