Angular momentum operator $\vec{J} = (J_1, J_2, J_3)$

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

One diagonalises $J^2 = \vec{J}^2$ and $J_z$.

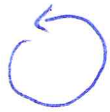
The basis is given by $|jm\rangle$ with $j = 0, \frac{1}{2}, 1, \dots$

$$\begin{cases} J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \\ J_z |jm\rangle = \hbar m |jm\rangle \end{cases}$$

$m = -j, \dots, j$.

Now, there are two types of angular momentum.

$\vec{L}$ $(= \vec{r} \times \vec{p})$



In this case $j = l = 0, 1, 2, \dots$ $\quad e^{im\varphi} = e^{im(\varphi + 2\pi)}$

But particles also have an intrinsic angular momentum, called spin:

$\vec{S}$

The spin $j = s = 0, \frac{1}{2}, 1, \dots$ can also be half-integer.

Integer: $s = 0, 1, 2, \dots$ bosons (Bose statistics → symmetric states

Half-integer: $s = \frac{1}{2}, \frac{3}{2}, \dots$ fermions (Fermi statistics → antisymmetric states

Elementary particles:

$s = 1$ photons, gluons, $W^{\pm}$ $Z^0$
$\gamma$

$s = 1/2$ $\begin{pmatrix} e^- \\ \nu_e \end{pmatrix}$ $\begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix}$ $\begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix}$ $\begin{pmatrix} u \\ d \end{pmatrix}$ $\begin{pmatrix} c \\ s \end{pmatrix}$ $\begin{pmatrix} t \\ b \end{pmatrix}$

$s = 0$ The Higgs (2012)

$s = 2$ The graviton (?)

other spin values could be obtained for non-elementary hadronic states (such as the bound states of $\bar{q}q$ or of gluons...

TRIVIAL EXAMPLE: zero spin

$S = 0$

$|0,0\rangle \equiv$ trivial ... no way to change something ...

FIRST NONTRIVIAL CASE: $S = 1/2$ (as the $e^-$)

$$\begin{cases} \overset{s}{} \; \overset{m}{} \\ |\frac{1}{2}, \frac{1}{2}\rangle \\ |\frac{1}{2}, -\frac{1}{2}\rangle \end{cases}$$

Very often the first entry is omitted ...

Then we write

$$|\tfrac{1}{2}\rangle = |+\rangle = |\uparrow\rangle$$

$$|-\tfrac{1}{2}\rangle = |-\rangle = |\downarrow\rangle$$

$$\begin{cases} S^2 |\frac{1}{2}, \pm\frac{1}{2}\rangle = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right) |\frac{1}{2}, \pm\frac{1}{2}\rangle \\ S_z |\frac{1}{2}, \frac{1}{2}\rangle = \frac{\hbar}{2} |\frac{1}{2}, \frac{1}{2}\rangle \\ S_- |\frac{1}{2}, -\frac{1}{2}\rangle = -\frac{\hbar}{2} |\frac{1}{2}, -\frac{1}{2}\rangle \end{cases}$$

Pauli matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These are 2×2 complex matrices which fulfill important mathematical relations.

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_2$$

$$\vec{\sigma}^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3 \cdot 1_2$$

$$\left\{ \begin{array}{l} [\sigma_i, \sigma_j] = 2 \varepsilon_{ijk} \sigma_k \\ \{\sigma_i, \sigma_j\} = 2 \delta_{ij} \end{array} \right.$$

⇒ very close to what we need...

However, strictly speaking, the Pauli matrices are simply matrices... they are <u>not</u> operators....

In order to make
$|\frac{1}{2},\frac{1}{2}\rangle = |+\rangle$, $|\frac{1}{2},-\frac{1}{2}\rangle = |-\rangle$.

$$\hat{\sigma}_z = \left(|+\rangle, |-\rangle\right) \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{\sigma_z} \begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix}$$

$$= |+\rangle\langle +| - |-\rangle\langle -|$$

$$\begin{cases} \hat{\sigma}_z |+\rangle = |+\rangle \\ \hat{\sigma}_z |-\rangle = -|-\rangle \end{cases}$$

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$

$$\begin{cases} \hat{S}_z |+\rangle = \frac{\hbar}{2} |+\rangle \\ \hat{S}_z |-\rangle = -\frac{\hbar}{2} |-\rangle \end{cases}$$

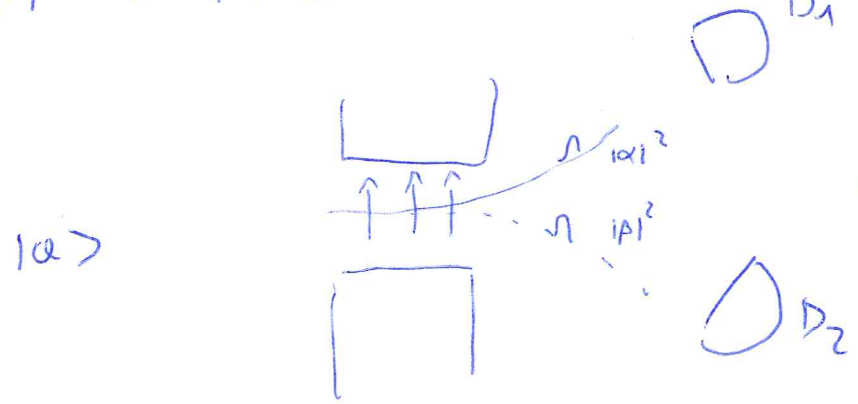
$|\varphi\rangle = \alpha|+\rangle + \beta|-\rangle$

Measure the spin and get:

$|\alpha|^2 \rightarrow$ spin "up".

$|\beta|^2 \rightarrow$ spin "down".

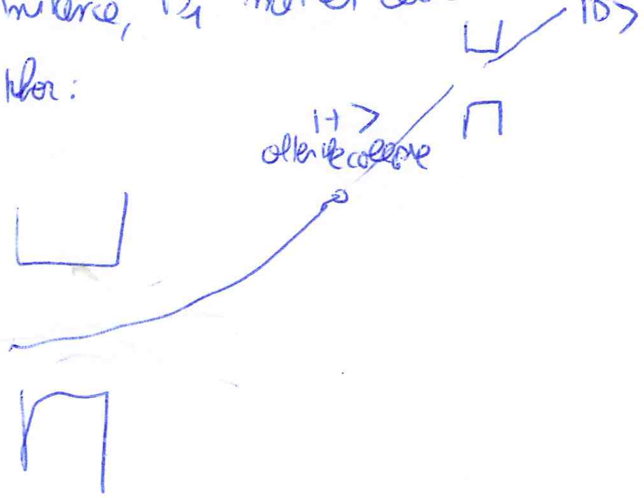
=




(wave-function is at the beginning a superposition...

stern-Gerlach-experiment.

For instance, $D_1$ makes click... let us suppose the particle goes further:




I will then measure $|+\rangle$ at each subsequent measurement.

What about $\hat{S}_x$ and $\hat{S}_y$?

$$\hat{\sigma}_x = (|+\rangle, |-\rangle)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix} = (|+\rangle, |-\rangle)\begin{pmatrix} \langle -| \\ \langle +| \end{pmatrix}$$

$$\boxed{\hat{\sigma}_x = |+\rangle\langle -| + |-\rangle\langle +|}$$

$$\begin{cases} \hat{\sigma}_x |+\rangle = |-\rangle \\ \hat{\sigma}_x |-\rangle = |+\rangle \end{cases}$$

$|+\rangle, |-\rangle$ are not eigenkets. One has an interchange of them....

$$\hat{S}_x = \frac{\hat{\sigma}_x}{2}$$

is the operator of the spin in the x-direction.

One is still missing: $\hat{S}_y$.

$$\hat{\sigma}_y = (|+\rangle, |-\rangle) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix} = (|+\rangle, |-\rangle) \begin{pmatrix} -i \langle -| \\ i \langle +| \end{pmatrix}$$

$$= -i\,|+\rangle\langle -| + i\,|-\rangle\langle +|$$

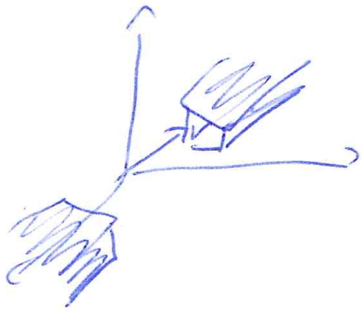
$$\begin{cases} \hat{\sigma}_y |+\rangle = +i\,|-\rangle, \\ \hat{\sigma}_y |-\rangle = -i\,|+\rangle . \end{cases}$$

it also interchanges $|+\rangle$ to $|-\rangle$ but with some "$i$" involved.

$$\hat{S}_y = \frac{\hat{\sigma}_y}{2} .$$

Exercise

$|\psi\rangle = |+\rangle \uparrow$



$|+\rangle$ goes in, but we measure the spin along the x-direction.

Which is the result?

We need to find the eigenvalues of $S_x$ and express $|\psi\rangle$ in that basis.

(Intuitively: 50% - 50%).

$$\begin{cases} |x,+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ |x,-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \end{cases}$$

are eigenstates of $\hat{\sigma}_x$ in

$$\hat{\sigma}_x |x,+\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle) = |x,+\rangle \qquad \text{(eigenvalue: +1)}$$

$$\hat{\sigma}_x |x,-\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle) = -|x,-\rangle \qquad \text{(eigenvalue: -1)}$$

$\hat{S}_x = \frac{\hbar\hat{\sigma}_x}{2} \longrightarrow |x,+\rangle$ has spin $\frac{\hbar}{2}$ in the x-direction

$|x,-\rangle$ " " $-\frac{\hbar}{2}$ " " " "

Invert:

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|x,+\rangle + |x,-\rangle\right)$$

if I measure $|+\rangle$ along the x-direction I find: $+\frac{\hbar}{2}$ with prob. 50% and $-\frac{\hbar}{2}$ with prob. +50%.

Generalise:

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \frac{\alpha}{\sqrt{2}}\left(|x,+\rangle + |x,-\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|x,+\rangle - |x,-\rangle\right)$$

$$= \left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right)|x,+\rangle + \left(\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right)|x,-\rangle$$

$\left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 \longrightarrow \frac{\hbar}{2}$ (x-direction)

$\left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 \longrightarrow -\frac{\hbar}{2}$ ( " " )

$$\boxed{\begin{aligned} \langle S_z\rangle_\psi &= |\alpha|^2\frac{\hbar}{2} - |\beta|^2\frac{\hbar}{2} \\ \langle S_x\rangle_\psi &= \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2\frac{\hbar}{2} - \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2\frac{\hbar}{2} \end{aligned}} \boxed{= \frac{\hbar}{2}\left(\alpha^*\beta + \beta^*\alpha\right).}$$

In some cases one uses a somewhat lighter notation, which makes use the basis $|+\rangle$ and $|-\rangle$ as privileged.

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \in \mathcal{F} \longmapsto \vec{\psi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 .$$

$$|+\rangle \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

All the previous relation are somewhat easier.

For instance:

$$\langle \sigma_x \rangle_\psi = \vec{\psi}^+ \sigma_x \psi = (\alpha^*, \beta^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha^* \beta + \beta^* \alpha .$$

$$\langle \sigma_z \rangle_\psi = \psi^+ \sigma_z \psi = (\alpha^*, \beta^*) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 - |\beta|^2 .$$

Also for higher spin one can repeat the analysis. For instance, in the case of the spin "1":

$|1,1\rangle \equiv |1\rangle$

$|1,0\rangle \equiv |0\rangle$

$|1,-1\rangle \equiv |-1\rangle$

one introduces the matrices

$$\Sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\left[\Sigma_i , \Sigma_j\right] = i\,\varepsilon_{ijk}\,\Sigma_k$$

$$S_i = \hbar\,\left(|1\rangle, |0\rangle, |-1\rangle\right)\Sigma_i \begin{pmatrix} \langle 1| \\ \langle 0| \\ \langle -1| \end{pmatrix}$$

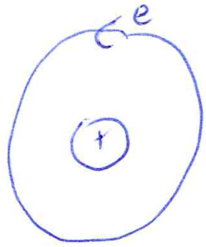
then all go as before. A generic state is:

$$|\psi\rangle = \alpha|1\rangle + \beta|0\rangle + \gamma|-1\rangle \equiv \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$q \quad \bar{q}$

$\frac{1}{2} \quad \frac{1}{2}$

Which is the total spin of the meson?



$e^-$ $\vec{s} = \frac{1}{2}$, $L$ → which is the total angular momentum?

In general, given two angular momenta $\vec{J}_1, \vec{J}_2$: how do we compose them? We assume that

The answer is "not" trivial.

The first thing is to write down the relation for $\vec{J}_1$ and $\vec{J}_2$

$$\begin{cases} \vec{J}_1^{\,2} |J_1 m_1\rangle = J_1(J_1+1)|J_1 m_1\rangle \\ J_2^2 |J_2 m_2\rangle = J_2(J_2+1)|J_2 m_2\rangle \end{cases}$$

Then:

$$J_{1,z} |J_1, m_1\rangle = m_1 |J_1, m_1\rangle \qquad m_1 = -J_1, \dots, J_1$$

$$J_{2,z} |J_2, m_2\rangle = m_2 |J_2, m_2\rangle \qquad m_2 = -J_2, \dots, J_2$$

Now, let us fix $\underline{J_1, J_2}$ (full spin = fixed angular, intrinsic).
(if L → "fixed").

$|m_1\rangle$

$|m_2\rangle$

Now, we can construct the states

$$|m_1, m_2\rangle$$

$m_1 = -J_1, \dots, J_1$

$m_2 = -J_2, \dots, J_2$

Total nr of vectors: $(2J_1+1)(2J_2+1)$.

This is OK, but these vectors are not eigenstates of

$\vec{J}^2$ with $\vec{J} = \vec{J}_1 + \vec{J}_2$.

One has to recast them in a clever way.

$$\bar{J}_{1z} \cdot \bar{J}_{2z} \, |+,+\rangle = \frac{\hbar}{2} \cdot \frac{\hbar}{2} \, |+,+\rangle$$

$$J_{1x} \cdot J_{2x} \, |+,+\rangle = \frac{\hbar^2}{4} \cdot |-,-\rangle$$

$$J_{1y} \cdot J_{2y} \, |+,+\rangle = -\frac{\hbar^2}{4} \, |-,-\rangle$$

Ergo:

$$\bar{J}_1 \cdot \bar{J}_2 \, |+,+\rangle = \frac{\hbar^2}{4} \, |+,+\rangle .$$

Summarizing:

$$\bar{J}^2 \, |+,+\rangle = \left[ \hbar^2 \cdot \frac{1}{2} \cdot \frac{3}{2} + \hbar^2 \cdot \frac{1}{2} \cdot \frac{3}{2} + \frac{2\hbar^2}{4} \right] |+,+\rangle$$

$$= \hbar^2 \, 1 \, (1+1) \, |+,+\rangle$$

$$\uparrow$$

$J_{tot} = 1$

The same for $|-,-\rangle$.

But not for $|+,-\rangle$ ...

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right)$ In nicht blatt $\vec{J}^2$ has still value 1.

This is because of the sign in between...

$$J_{1z} \cdot J_{2,z} |\psi\rangle = -\frac{\hbar^2}{4} |\psi\rangle$$

$$J_{1,x} \cdot J_{2,x} |\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{\hbar^2}{4}|-,+\rangle + \frac{\hbar^2}{4}|+,-\rangle\right) = \frac{\hbar^2}{4}|\psi\rangle$$

$$J_{1,y} \cdot J_{2,y} |\psi\rangle = \frac{1}{\sqrt{2}}\left(+|-,+\rangle + |+,-\rangle\right) = \frac{\hbar^2}{4}|\psi\rangle$$

$$\bar{J}_1 \cdot \bar{J}_2 |\psi\rangle = \frac{\hbar^2}{4} |\psi\rangle$$

Put all together:

$$(\bar{J}_1 + \bar{J}_2)^2 |\psi\rangle = \left[\hbar^2 \frac{1}{2}\frac{3}{2} + \hbar^2 \frac{1}{2}\frac{3}{2} + 2\frac{\hbar^2}{4}\right] |\psi\rangle \; !$$

On the contrary if we study:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

we find:

$$J_{1k} J_{2k} |\psi\rangle = -\frac{\hbar^2}{4} |\psi\rangle \quad \forall k\,!$$

Ergo:

$$\left(\vec{J}_1 + \vec{J}_2\right)^2 |\psi\rangle = \hbar^2 \cdot \left[\frac{1}{2}\cdot\frac{3}{2} + \frac{1}{2}\cdot\frac{3}{2} + 2\cdot\left(-\frac{3}{4}\right)\right] = 0\,!!!$$

Summary:

$J=1$ with

TRIPLET

$$\begin{cases} |J=1, m=1\rangle = |+,+\rangle \\ |J=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\underset{\uparrow}{+},\underset{\downarrow}{-}\rangle + |\underset{\downarrow}{-},\underset{\uparrow}{+}\rangle\right) \\ |J=1, m=-1\rangle = | \end{cases}$$

$$\vec{J}^2 |J=1, m\rangle = \hbar^2 \cdot 1 \cdot (1+1) |J=1, m\rangle \quad \forall m$$

$$J_z |J=1, m\rangle = \hbar\, m\, |J=1, m\rangle$$

NOTE: SYMMETRIC

$J=0$ $\qquad |J=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$

SINGLET

$\vec{J}^2 |J=0, m=0\rangle = 0$

$J_z |J=0, m=0\rangle = 0$ NOTE: ANTISYMMETRIC.

Comparison between $|J=0, m=0\rangle$ and $|J=1, m=0\rangle$. Only a sign $\neq$ is present but that is crucial. it transform a state with spin 1 to a state with spin 0.

The symmetry is also reversed.

In general, out of $\bar{J}_1$ and $\bar{J}_2$

$(J_1, m_1)$ $(J_2, m_2)$

I can construct the basis $|m_1, m_2\rangle$ with $(2J_1+1)\cdot(2J_2+1)$ elements.

<u>But</u> I should then recount it differently.

$$\vec{J}_{tot} = \bar{J}_1 + \bar{J}_2$$

$$\begin{cases} J = |J_1 - J_2|, \dots, J_1 + J_2 \\ m = -J, \dots, J \end{cases}$$

$$|J_1 + J_2, m = J_1 + J_2\rangle = |m_1 = J_1, m_2 = J_2\rangle$$

$\dots$

Then I should construct all of them... smaller lengths once.

Example:

$\vec{L}$ with $l=1$, $\vec{S}$ with $s=1$. Basis: $|1,1\rangle, |1,0\rangle, \dots$

$\vec{J} = \vec{L} + \vec{S}$

$J = 2$

(symmetric)

$$
\begin{cases}
|J=2, m=2\rangle = |1\,1\rangle \\
|J=2, m=1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle) \\
|J=2, m=0\rangle = \alpha\left[\frac{1}{\sqrt{2}}(|1,-1\rangle + |-1,1\rangle)\right] + \beta|0,0\rangle \\
|J=2, m=-1\rangle = \frac{1}{\sqrt{2}}(|-1,0\rangle + |0,-1\rangle) \\
\\
|J=2, m=-2\rangle = |-1,-1\rangle
\end{cases}
$$

$J = 1$

(antisymmetric)

$$
\begin{cases}
|J=1, m=1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle) \\
|J=1, m=0\rangle = \frac{1}{\sqrt{2}}(|1,-1\rangle - |-1,1\rangle) \\
|J=1, m=-1\rangle = \frac{1}{\sqrt{2}}(|-1,0\rangle - |0,-1\rangle
\end{cases}
$$

$J = 0$

(symmetric)

$|J=0, m=0\rangle = \perp$ to $|J=2, m=0\rangle$

$\alpha$:

Check:

$$\boxed{\Sigma_z |1\rangle = |1\rangle \qquad \Sigma_z |0\rangle = 0 \qquad \Sigma_z |-1\rangle = -|1\rangle}$$

$$\Sigma_x |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{\Sigma_x |1\rangle = \frac{1}{\sqrt{2}} |0\rangle}$$

$$\Sigma_x |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left( |1\rangle + |-1\rangle \right)$$

$$\boxed{\Sigma_x |0\rangle = \frac{1}{\sqrt{2}} \left( |1\rangle + |-1\rangle \right)}$$

$$\boxed{\Sigma_x |-1\rangle = \frac{1}{\sqrt{2}} |0\rangle} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Sigma_y |1\rangle = \frac{i}{\sqrt{2}} |0\rangle$$

$$\Sigma_y |0\rangle = \frac{i}{\sqrt{2}} \left( -|1\rangle + |-1\rangle \right) \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{+i}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Sigma_y |-1\rangle = \frac{-i}{\sqrt{2}} |0\rangle \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}$$

check the state $|0,0\rangle$

$$\left(\Sigma_1^2 + \Sigma_2^2 + 2\vec{\Sigma}_1\vec{\Sigma}_2\right)|0,0\rangle =$$

$$= \left[1(1+1) + 1(1+1)\right]|0,0\rangle + 2\vec{\Sigma}_1\vec{\Sigma}_2|0,0\rangle$$

$$\Sigma_{1x}\Sigma_{2x}|0,0\rangle = \left(\frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)\right)\frac{1}{\sqrt{2}}\left(+|1\rangle + |-1\rangle\right)$$

$$\Sigma_{1y}\Sigma_{2y}|0,0\rangle = \left(\frac{i}{\sqrt{2}}(-|1\rangle + |-1\rangle)\right)\left(\frac{i}{\sqrt{2}}\left(-|+1\rangle + |-1\rangle\right)\right)$$

Definitely not an eigenstate !!!

$$\begin{cases} \Sigma_{1x}\Sigma_{2x}|1,-1\rangle = \frac{1}{2}|0,0\rangle \\ \Sigma_{1y}\Sigma_{2y}|1,-1\rangle = \frac{1}{2}|0,0\rangle \end{cases}$$

But:

$$\left(\Sigma_{1x}\Sigma_{2x} + \Sigma_{1y}\Sigma_{2y}\right)|0,0\rangle = \frac{1}{2}\left(|1,1\rangle + |1,-1\rangle + |-1,1\rangle + |-1,-1\rangle\right)$$

$$+ \frac{-1}{2}\left(|1,1\rangle - |1,-1\rangle - |-1,1\rangle + |-1,-1\rangle\right) =$$

$|1,-1\rangle + |-1,1\rangle$

Ergo $|0,0\rangle$ is not an eigentable....

$$|\psi\rangle = \alpha\left[\frac{1}{\sqrt{2}}\left(|1,-1\rangle + |-1,1\rangle\right)\right] + \beta|0,0\rangle$$

$$\Sigma_{1z}\Sigma_{2z}|\psi\rangle = -\alpha\left[\frac{1}{\sqrt{2}}\left(|1,-1\rangle + |-1,1\rangle\right)\right] -$$

$$\Sigma_{1x}\Sigma_{2x}|\psi\rangle = \alpha\left[\frac{1}{\sqrt{2}}\left(\frac{1}{2}|0,0\rangle + \frac{1}{2}|0,0\rangle\right)\right] + \beta\left(\Sigma_{1x}\Sigma_{2x}|0,0\rangle\right.$$

$$\Sigma_{1y}\Sigma_{2y}|\psi\rangle = \alpha\left[\frac{1}{\sqrt{2}}\left(\frac{1}{2}|0,0\rangle + \frac{1}{2}|0,0\rangle\right)\right] + \beta\Sigma_{1y}\Sigma_{2y}|0,0\rangle$$

Sum:

$$\left(\Sigma_{1z}\Sigma_{2z} + \Sigma_{1x}\Sigma_{2x} + \Sigma_{1y}\Sigma_{2y}\right)|\psi\rangle =$$

$$= -\alpha\frac{1}{\sqrt{2}}\left(|1,-1\rangle + |-1,1\rangle\right) + \frac{\alpha}{\sqrt{2}}\left(|0,0\rangle\right) + \frac{\alpha}{\sqrt{2}}\left(|0,0\rangle\right)$$

$$+ \beta\left[|1,-1\rangle + |-1,1\rangle\right]$$

$$= \left(\sqrt{2}\beta - \alpha\right)\left[\frac{1}{\sqrt{2}}\left(|1,-1\rangle + |-1,1\rangle\right)\right] + \sqrt{2}\alpha|0,0\rangle$$

$$\begin{cases} \sqrt{2}\beta - \alpha = \lambda\alpha \\ \sqrt{2}\alpha = \lambda\beta \end{cases} \qquad |\alpha|^2 + |\beta|^2 = 1.$$

$$\alpha = \frac{\lambda}{\sqrt{2}}\beta$$

$$\sqrt{2}\cdot\beta - \frac{\lambda}{\sqrt{2}}\beta = \lambda\cdot\frac{\lambda}{\sqrt{2}}\beta$$

$$\lambda^2\cdot\frac{\beta}{\sqrt{2}} + \lambda\cdot\frac{\beta}{\sqrt{2}} - \sqrt{2}\beta \quad \rightarrow \text{Eliminate } \beta$$

$$\frac{\lambda^2}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}} - \sqrt{2} = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4\cdot(-2)}}{2} = \frac{-1\pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$\boxed{\lambda = 1}$$

$\boxed{\lambda = -2}$ In the other case...

Ergo "almost done":

$\lambda = 1$

$\alpha = \beta = \frac{1}{\sqrt{2}}$ ✓

$\lambda = -2$

$\sqrt{2}\,\alpha = -2\beta$

$\alpha = -\frac{2}{\sqrt{2}}\beta = -\sqrt{2}\,\beta$

$\alpha^2 + \beta^2 = 1 \qquad 2\beta^2 + \beta^2 = 1 \qquad \beta = \frac{1}{\sqrt{3}}$

$\alpha = \sqrt{\frac{2}{3}}$