Hilbert space $\mathcal{F}$.

$|f\rangle \in \mathcal{F}$; it describes a physical state if $\langle f|f\rangle = 1$.

it corresponds univocally to a function $f(\vec{x}) \in L^2$.

In this respect:

$\{|\vec{x}\rangle\}$ / $\langle \bar{x}_1|\bar{x}_2\rangle = \delta(\bar{x}_1 - \bar{x}_2)$

$\langle \bar{x}|f\rangle = f(\bar{x})$

$$(|f_1\rangle, |f_2\rangle) \; / \; \begin{cases} \langle \bar{x}|f_1\rangle = f_1(\bar{x}) \\ \langle \bar{x}|f_2\rangle = f_2(\bar{x}) \end{cases}$$

$$\langle f_1|f_2\rangle = (f_1, f_2) = \int d^3x \, f_1^*(\bar{x}) f_2(\bar{x})$$

basis: $\{|\varphi_n\rangle\}$ $\quad \langle \varphi_n|\varphi_m\rangle = \delta_{nm}$

$$\sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n| = 1$$

Recall: the bra $\langle f_1|$ can be seen as an operator $\mathcal{F} \mapsto \mathbb{C}$

$$\langle f_1| \big(|f_2\rangle\big) = \langle f_1|f_2\rangle$$

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$\{|\vec{x}\rangle\}$ is indeed a basis with a countless number of elements. This is always a good basis for 1-particle QM.

$\langle \vec{x}_1 | \vec{x}_2 \rangle = \delta(\vec{x}_1 - \vec{x}_2)$

$\int d^3x \, |\vec{x}\rangle\langle\vec{x}| = 1$

$\{|\varphi_n\rangle\}$ such that $H|\varphi_n\rangle = E_n|\varphi_n\rangle$ is a basis with a countable nr of states.

it is not always possible to have such a basis.

Harmonic oscillator: yes. Binding potential: yes.

$V(\vec{x}) = 0 \Rightarrow$ NO. Non-binding potential: No.

H-atom: also NO (mixed spectrum).

$|\vec{k}\rangle$ / $\langle \vec{k} | \vec{x} \rangle = \dfrac{e^{i\vec{k}\vec{x}}}{(2\pi)^{3/2}}$

$$\begin{cases} \langle \vec{k}_1 | \vec{k}_2 \rangle = \delta^3(\vec{k}_1 - \vec{k}_2) \\ \int d^3k \, |\vec{k}\rangle\langle\vec{k}| = 1 \end{cases}$$

A generic state $|f\rangle$ with $\langle\bar{x}|f\rangle = f(\bar{x})$ and $\langle\bar{k}|f\rangle = A(\bar{k})$ can be written as:

$$|f\rangle = \left(\int d^3x\, |\bar{x}\rangle\langle\bar{x}|\right)|f\rangle = \int d^3x\, |\bar{x}\rangle \underbrace{\langle\bar{x}|f\rangle}_{f(\bar{x})} = \int d^3x\, f(\bar{x})\,|\bar{x}\rangle =$$

$$= \int d^3k\, |\bar{k}\rangle\langle\bar{k}|f\rangle = \int d^3k\, A(\bar{k})\,|\bar{k}\rangle$$

$$A(\bar{k}) = \int \frac{d^3x}{(2\pi)^{3/2}} f(\bar{x})\, e^{i\bar{k}\bar{x}} \qquad A(\bar{k}) \xleftrightarrow{\text{Fourier transf.}} f(\bar{x})$$

$\Rightarrow$ operator $A_{\bar{x}} : L^2 \mapsto L^2$

$$A_{\bar{x}} f(\bar{x}) = g(\bar{x})$$

A is the corresp. operator $\mathcal{F} \mapsto \mathcal{F}$ such that

$$A|f\rangle = |g\rangle$$

Moreover

$$\underset{\text{in } L^2}{[A_{\bar{x}}, B_{\bar{x}}] = C_{\bar{x}}} \quad \Rightarrow \quad \underset{\text{in } \mathcal{F}}{[A, B] = C}$$

$[X, P_x] = i\hbar$ in both cases

Schrödinger eq. in $\mathcal{F}$:

$|\psi(t)\rangle$ such that $\langle \vec{x} | \psi(t)\rangle = \psi(t, \vec{x})$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi_0\rangle \quad \text{with } |\psi_0\rangle = |\psi(t=0)\rangle$$

We are "tacitly" working in the so-called "Schrödinger representation".

The states evolve with time through the Schrödinger eq.

The operators are usually time-independent $(\hat{A})$.

(They may have an explicit dependence, for instance

I construct

$\hat{A}(t) = t\hat{H}$.

But this is not usual ...)

An important side remark:

$$(|f\rangle)^{\dagger} = \langle f|$$

$$A = c\,|f_1\rangle\langle f_2|$$ This is an operator: $\mathcal{F} \mapsto \mathcal{F}$.
is it physical?

$$A^{\dagger} = c^{*}\,|f_2\rangle\langle f_1| \neq A$$ $\rightarrow$ No, because it is not herm.

This is not hermitian...

A possible hermitian operator is:

$$\boxed{A = c\,|f_1\rangle\langle f_2| + c^{*}\,|f_2\rangle\langle f_1|}$$

Example:

$\{|a\rangle, |b\rangle\}$ is a basis.

$\langle a|a\rangle = \langle b|b\rangle = 1$, $\langle a|b\rangle = \langle b|a\rangle = 0$.

$H = \alpha|a\rangle\langle a| + \beta|b\rangle\langle b| + \gamma(|a\rangle\langle b| + |b\rangle\langle a|)$

it is herm. if α, β, γ are real nos.

Namely, in the present form:

$H^\dagger = \alpha^*|a\rangle\langle a| + \beta^*|b\rangle\langle b| + \gamma^*(|b\rangle\langle a| + |a\rangle\langle b|)$

$= H$ if $\begin{cases} \alpha^* = \alpha \\ \beta^* = \beta \\ \gamma^* = \gamma \end{cases}$

$|a\rangle$ is not an eigenstate of H:

$H|a\rangle = \alpha|a\rangle + \gamma|b\rangle \not\propto |a\rangle$.

$|a\rangle$ is not an eigenstate of H.

$H|b\rangle = \beta|b\rangle + \gamma|a\rangle$

$|b\rangle$ is not an eigenstate of (H).

The term proportional to γ "mixes" them... it transforms one into the other.

If, for instance, $|\psi_0\rangle = |\psi(0)\rangle = |a\rangle$, how does the system evolve in time? $|a\rangle$ is not an eigenvalue of H, therefore

$$|\psi(t)\rangle = e^{-iHt/\hbar}|a\rangle$$

cannot be easily calculated. (This is possible only for $\gamma = 0$, for which $H|a\rangle = \gamma|a\rangle$)

We need to find the eigenstates of H.

$c = \cos\theta$
$s = \sin\theta$

$$\begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} |a\rangle \\ |b\rangle \end{pmatrix}$$

$$\begin{pmatrix} |a\rangle \\ |b\rangle \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \langle a| \\ \langle b| \end{pmatrix}$$

$$\begin{pmatrix} \langle a| \\ \langle b| \end{pmatrix} = \begin{pmatrix} c & -s \\ +s & c \end{pmatrix} \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}$$

Now it is "just" algebra. Plug it in:

$$H = \alpha \overset{|a\rangle}{\left(c|1\rangle - s|2\rangle\right)} \overset{\langle a|}{\left(c\langle 1| - s\langle 2|\right)} + \beta \left(+s|1\rangle + c|2\rangle\right)\left(+s\langle 1| + c\langle 2|\right)$$

$$+ \gamma \left(\left(c|1\rangle - s|2\rangle\right)\left(s\langle 1| + c\langle 2|\right) + \left(s|1\rangle + c|2\rangle\right)\left(c\langle 1| - s\langle 2|\right) \right)$$

$$= |1\rangle\langle 1| \left(\alpha c^2 + \beta s^2 + 2\gamma s c \right)$$

$$+ |2\rangle\langle 2| \left(\alpha s^2 + \beta c^2 - 2\gamma s c \right)$$

$$+ |1\rangle\langle 2| \left(-\alpha s c + \beta s c + \gamma (c^2 - s^2) \right) \quad + |2\rangle\langle 1| \left(\dots \right)$$

If we want that $|1\rangle$ and $|2\rangle$ are eigenstates:

$$-\alpha s c + \beta s c + \gamma (c^2 - s^2) = 0$$

$$\gamma \cos(2\theta) = (\alpha - \beta) s c = \frac{(\alpha - \beta)}{2} \sin(2\theta)$$

$$\tan(2\theta) = \frac{2\gamma}{\alpha - \beta}$$

$$\theta = \frac{1}{2} \arctan\left[\frac{2\gamma}{\alpha - \beta} \right]$$

$$E_1 = \alpha c^2 + \beta s^2 + 2\gamma s c$$

$$E_2 = \alpha s^2 + \beta c^2 - 2\gamma s c$$

Ergo:

$$H = E_1 |1\rangle\langle 1| + E_2 |E_2\rangle\langle E_2|$$

$$e^{-iHt/\hbar}|a\rangle = e^{-iHt/\hbar}\left(c|1\rangle - s|2\rangle\right) =$$

$$= e^{-iE_1 t/\hbar} c|1\rangle - e^{-iE_2 t/\hbar} s|2\rangle =$$

$$= |\psi(t)\rangle \;!!!$$

We solved a mixing problem.

[E_1, E_2, can be also obtained by diag.

the matrix $\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} = M$

E_1, E_2 are the eigenvalues of M.

$|1\rangle, |2\rangle$ are the eigenvectors.]

This mixing problem is a basic problem of physics. Mixing of elementary particles (gluons - quarks), non-diagonal hamiltonians to be diag., etc etc...

Namely:

$$H = (|a\rangle, |b\rangle) \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} \langle a| \\ \langle b| \end{pmatrix}$$

$$\begin{pmatrix} \langle a| \\ \langle b| \end{pmatrix} = B \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}$$

$$(|a\rangle, |b\rangle) = (|1\rangle, |2\rangle) B^{t}$$

$$B^{t} M B = D.$$

$\hat{A}$, $|\psi(t)\rangle$ with $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$.

The average of $\hat{A}$ for a particle described by $|\psi(t)\rangle$ is

$$\langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\hat{A}\rangle_{\psi}$$

$\hat{A}$ is time-indep.

$|\psi(t)\rangle$ time dep.

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle \qquad |\psi_0\rangle = |\psi(t=0)\rangle$$

Ergo:

$$\langle A\rangle_{\psi} = \langle\psi(t)|\hat{A}|\psi(t)\rangle$$

$$= \langle\psi_0|$$

Namely:

$$\langle\psi(t)| = \langle\psi_0|e^{+iHt/\hbar}$$

$$\begin{pmatrix} |a\rangle = A|b\rangle \\ \langle a| = \langle b|A^{+} \end{pmatrix}$$

$$\langle \hat{A} \rangle = \langle \psi_0 | e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} | \psi_0 \rangle$$

Heisenberg picture:

The state is time-independent: $|\psi\rangle_H = |\psi_0\rangle = |\psi(t=0)\rangle$

The operators are time-dependent:

$$\hat{A}_H^{(t)} = e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar}$$

$$\frac{d\hat{A}_H}{dt} = \frac{i\hat{H}}{\hbar}\left(e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}\right) + \left(e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}\right)\left(\frac{-i\hat{H}}{\hbar}\right)$$

Ergo:

$$\boxed{\frac{d\hat{A}_H}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}_H]}$$

Heisenberg e.o.m.

This is the eq. of motion in the Heisenberg picture.

Note:

$\hat{H}_H = \hat{H}$ still indep. on t; and: $\frac{d\hat{H}_H}{dt} = \frac{d\hat{H}}{dt}$

[if $\hat{A}$ has also an explicit time dependence:

$$\frac{d\hat{A}_H}{dt} = \frac{\partial \hat{A}_H}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{A}_H]$$

This approach is formally very similar to Poisson-brackets.

Consequence:
the operator $\hat{A}_H$ is a constant of motion if $[\hat{H}, \hat{A}_H] = 0$.

Harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

We already know that:

$$H|n\rangle = E_n |n\rangle \qquad \left(\begin{aligned} &E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \\ &\langle x|n\rangle = \psi_n(x) = N_n H_n(x) e^{-Ax^2} \end{aligned} \right)$$

Define:

$$a = \frac{1}{\sqrt{2m\hbar\omega}} (p - im\omega x)$$

$$\left\{ \begin{aligned} &[x,x] = 0 \\ &[x,p] = i\hbar \\ &[p,p] = 0 \end{aligned} \right. \longrightarrow \left\{ \begin{aligned} &[a,a] = [a^\dagger, a^\dagger] = 0 \\ &[a, a^\dagger] = 1 \end{aligned} \right. \Longrightarrow$$

see next page

$|0\rangle$ is the state with the minimal energy.

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$[a, a^\dagger] = \frac{1}{2m\hbar\omega}\left[p - im\omega x, p + im\omega x\right] =$$

$$= \frac{1}{2m\hbar\omega}\left(im\omega[p, x] + [-im\omega x, p]\right) =$$

$$= \frac{1}{2m\hbar\omega}(-im\omega \cdot 2 \cdot i\hbar) = 1 \; !!!$$

Exercise:

$[H, a] = ?$

$$\left[\hbar\omega\left(a^{+}a+\tfrac{1}{2}\right), a\right] = \hbar\omega\,[a^{+}a, a] =$$

$$= \hbar\omega\, a^{+}\underbrace{[a,a]}_{=0} + \hbar\omega\underbrace{[a^{+},a]}_{-1}a =$$

$$= -\hbar\omega a.$$

$[H, a] = -\hbar\omega a$

Consequence. $H|n\rangle = E_n|n\rangle$

$a|n\rangle$

$$H(a|n\rangle) = (aH - \hbar\omega a)|n\rangle =$$

$$= a(H-\hbar\omega)|n\rangle = a(E_n - \hbar\omega)|n\rangle$$

$$= (E_n - \hbar\omega)a|n\rangle$$

Ergo:

$a|n\rangle$ is also an eigenstate of H; its energy is $(E_n - \hbar\omega)$.

$a|n\rangle = \# |n-1\rangle$

$a^2|n\rangle = \# |n-2\rangle$

...

$a^2|n\rangle$ has energy $(E_n - 2\hbar\omega)$, etc.

Then, in order to have an energy bounded from below,

$a|0\rangle = 0$

But then: $H|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$.

$|0\rangle$ has the energy of $\frac{1}{2}\hbar\omega$ = ground state.

$[H, a^\dagger] = \hbar\omega a^\dagger$

$a^\dagger|n\rangle$ has energy $E_n + \hbar\omega$.

Ergo, one can construct the various states starting by $|0\rangle$.

$a^\dagger|0\rangle$ has energy $\frac{1}{2}\hbar\omega + \hbar\omega = \frac{3}{2}\hbar\omega$

$|1\rangle = a^\dagger|0\rangle$

Normalized? Yes. $\langle 1| = \langle 0|a$

$\langle 1|1\rangle = \langle 0|a a^\dagger|0\rangle =$

$a^{+2}|0\rangle$ has energy $2\hbar\omega + \frac{1}{2}\hbar\omega = \frac{3}{2}\hbar\omega$.

How, $|S\rangle$ is not normalised.

$|S\rangle = a^+ a^+ |0\rangle$

$\langle S| = \langle 0| a a$

$\langle S|S\rangle = \langle 0|a a a^+ a^+|0\rangle =$

$= \langle 0| a(a^+ a + 1) a^+|0\rangle =$

$= \langle 0|a a^+ a a^+|0\rangle + \underbrace{\langle 0|a a^+|0\rangle}_{1} =$

$= \langle 0|(a^+ a + 1) a a^+|0\rangle + 1 =$

$= 0 + 1 + 1 = 2$

Ergo:

$|2\rangle = \frac{1}{\sqrt{2}}(a^+)^2|0\rangle$