Limit

(Limes (Grenzwert))

$f(x): D \subseteq R \mapsto R$ wobei D: Teilmenge von R

$\lim_{x \mapsto x_0} f(x) = L$

"f(x) hat für x gegen x_0 den Limes L,
wenn es zu jedem (noch so kleinen) $\varepsilon > 0$
ein (im Allgemeinen von ε abhängiges) $\delta > 0$ gibt,
sodass $\forall x \in D$ die
die Bedingung $|x - x_0| < \delta$ genügen,
auch $|f(x) - L| < \varepsilon$ gilt."

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

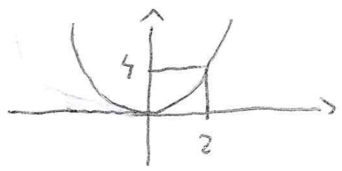
*) The concept of limit is crucial in mathematics, being one of the basic concepts on which all the rest is built.

*) This is a formalisation of a very intuitive idea: what happens if a variable tends to a particular value.

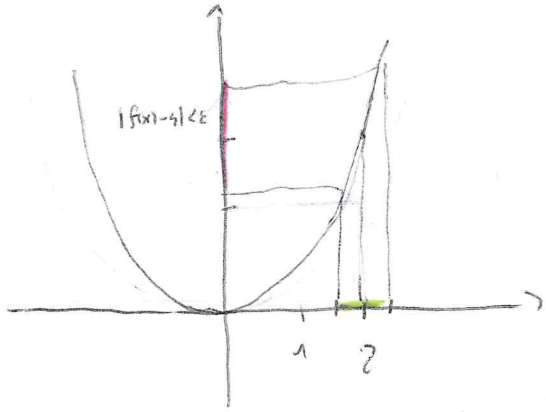
*) Generalisation of the definition for $x \mapsto \pm\infty$ and $L \mapsto \pm\infty$.

Example 1:

$$\lim_{x \to 2} x^2 = 4$$



"actually trivial ..." but let us show that the definition works.



intuitive correspondence

if $|x-2| < \delta \to |x^2-4| = |(x-2)(x+2)| = |x-2|\,|x+2| < \delta |x+2| = \delta |x-2+4|$

$$< \delta\big(|x-2|+4\big) < \delta(\delta+4) = \delta^2 + 4\delta = \varepsilon$$

$|x^2-4| < \delta^2 + 4\delta = \varepsilon \to \delta^2 + 4\delta - \varepsilon = 0$

Ergo: $\delta = -2 \pm \sqrt{4+\varepsilon} \to \delta \doteq \delta(\varepsilon) = -2 + \sqrt{4+\varepsilon}$ $\to$ Not so "easy" indeed ...

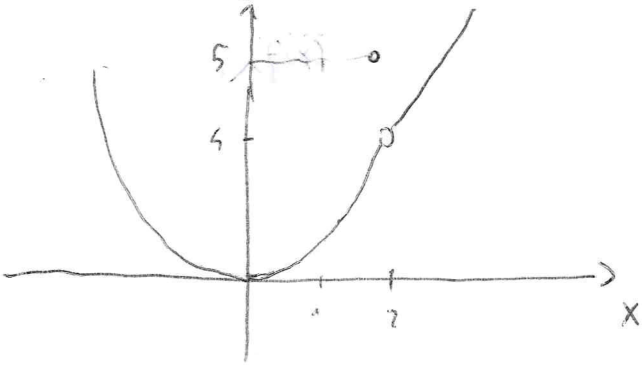
Summarizing, when closing

$$|x-2| < \underbrace{-2 + \sqrt{4+\varepsilon}}_{\delta} \implies |x^2-4| < \varepsilon .$$

Example 2:

Let us define the function

$$f(x) = \begin{cases} x^2 & \text{for } x \neq 2 \\ 5 & \text{for } x = 2 \end{cases}$$



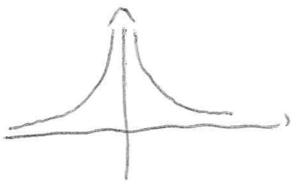
What is $\lim_{x \mapsto 2} f(x)$ in this case?

The limit is still 4! $\lim_{x \mapsto 2} f(x) = 4$. In fact, the limit is <u>not</u> always the value of the function in that point, but the value of the function when x is "tending" to that point.

Example 3:

$$f(x) = \frac{1}{x^2} : D \subseteq \mathbb{R} \mapsto \mathbb{R} \qquad D = (-\infty, 0) \cup (0, \infty)$$

($x_0 = 0$ is not part of D)



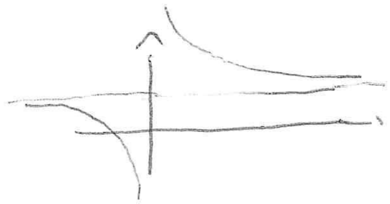
$$\lim_{x \mapsto 0} f(x) = +\infty$$

$$\left[\lim_{x \mapsto x_0} f(x) = +\infty \right.$$

$$\left. \forall N \text{ (very large)} > 0 \ \exists \ \delta > 0 \ / \ 0 < |x - x_0| < \delta \rightarrow f(x) > N \right]$$

Example 4:

$f(x) = \frac{1}{x} + 1$



$\lim_{x \to +\infty} f(x) = 1$

In general: $\lim_{x \to +\infty} f(x) = L$

$\forall \varepsilon > 0 \ \exists N > 0 \ / \ x > N \rightarrow |f(x) - L| < \varepsilon$

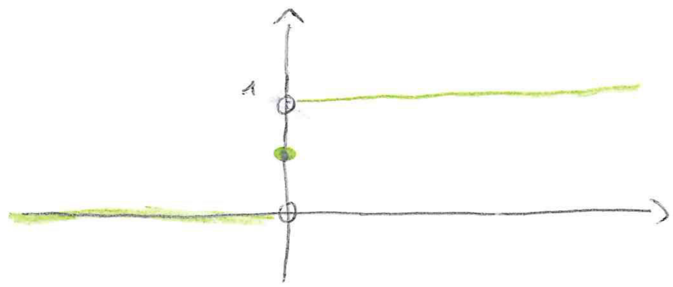
similarly:

$\lim_{x \to -\infty} f(x) = 1$

Example 5:

$$\vartheta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

'Heaviside' step function



$\lim_{x \to 0} \vartheta(x)$ is not defined.

How, if I come from the 'left':

$\lim_{x \to 0^-} \vartheta(x) = 0$

and, if I come from the right

$\lim_{x \to 0^+} \vartheta(x) = 1$

$\lim_{x \to x_0^-} f(x) = L$ $\quad \forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - x_0| < \delta$ and $x < x_0 \rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to x_0^+} f(x) = L$ $\quad \forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - x_0| < \delta$ and $x > x_0 \rightarrow |f(x) - L| < \varepsilon$

Derivability

def:

$f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$; $x_0 \in D$

$f(x)$ is derivable in x_0 if the limit $\lim_{h \mapsto 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists.

One writes:

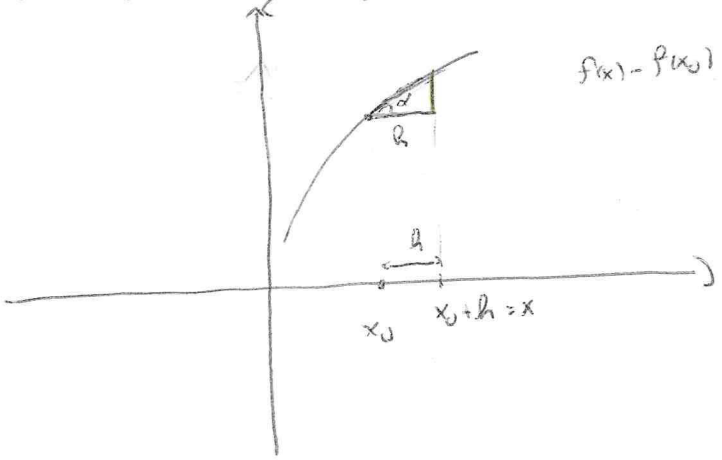
$$\lim_{h \mapsto 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) = \left(\frac{df}{dx}\right)_{x=x_0}$$

Note that, writing $x_0 + h = x$

$$f'(x_0) = \lim_{x \mapsto x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Geometrical meaning: $f'(x_0) = \tan\alpha$ (tangent to the function in x_0)

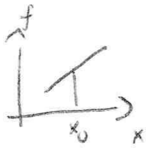
$f(x) - f(x_0) \simeq \tan\alpha \cdot (x - x_0)$ for x very close to x



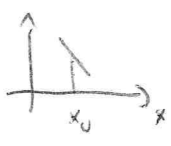
For x very close to x_0:

$f(x) \simeq f'(x_0) \cdot (x - x_0)$;

$f'(x_0) > 0$



$f'(x_0) < 0$



Example 1: derivative of x^2. $\quad f(x) = x^2$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^2 - x_0^2}{h} =$$

$$= \lim_{h \to 0} \frac{x_0^2 + h^2 + 2x_0 h - x_0^2}{h} = \lim_{h \to 0} \frac{h^2 + 2x_0 h}{h} =$$

$$= \lim_{h \to 0} (h + 2x_0) = 2x_0 .$$

In general, we write: $\boxed{f'(x) = 2x.}$

$$\left(\text{Generalization: } f(x) = x^n \mapsto f'(x) = n x^{n-1} \right).$$

Example 2: $\quad f(x) = \sin x$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} =$$

$$= \lim_{h \to 0} \sin x \cdot \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} + \lim_{h \to 0} \cos x \cdot \underbrace{\frac{\sin h}{h}}_{1} = \cos x.$$

$$\frac{d}{dx}(\sin x) = \cos x .$$

Similarly, one can evaluate the derivatives of all elementary functions.

Theorem: $f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$; $x_0 \in D$; $f(x)$ derivable in $x_0 \Rightarrow f(x)$ is continuous in x_0.

Per hypothesis the limit

$$\lim_{x \mapsto x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists and is well defined. Then, for $x \simeq x_0$ $\quad f(x) - f(x_0) = f'(x_0) \cdot (x - x_0)$.

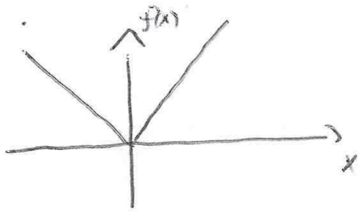
This means that:

$$\lim_{x \mapsto x_0} \left(f(x) - f(x_0) \right) = \lim_{x \mapsto 0} f'(x_0)(x - x_0) = 0 \Rightarrow \underline{\underline{\lim_{x \mapsto x_0} f(x) = f(x_0)}}.$$

The last term is the definition of continuity in x_0. q.e.d.

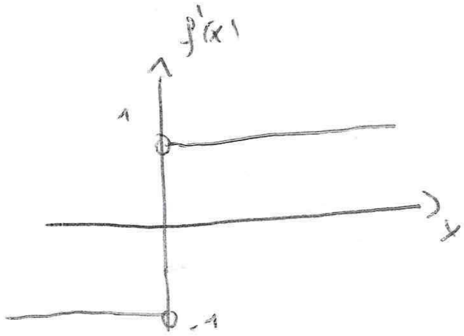
$$\boxed{f(x) \text{ derivable in } x_0 \rightarrow f(x) \text{ continuous in } x_0.}$$

ACHTUNG: the opposite is <u>not</u> true!



$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

$$\lim_{x \mapsto 0} f(x) = f(0) = 0.$$



<u>But</u>: $\lim_{h \mapsto 0} \frac{f(h) - f(0)}{h}$ does <u>not</u> exist!

(This is, btw, the sign-function)

Taylor series around the point $x_0 = 0$ (also called McLaurin)

$f(x) : D \subseteq \mathbb{R} \mapsto \mathbb{R}$; $x_0 = 0 \in D$.

We write the function $f(x)$ as a sum of polynomes:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

What are the numbers $a_0, a_1, a_2, \dots$?

$f(0) = a_0 + a_1 \cdot 0 + \dots = a_0 \quad \Rightarrow \quad a_0 = f(0)$.

$$\begin{cases} f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \\ f'(0) = a_1 \end{cases} \Rightarrow a_1 = f'(0).$$

$$\begin{cases} f''(x) = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \dots \\ f''(0) = 2a_2 \end{cases} \Rightarrow a_2 = \frac{1}{2} f''(0).$$

$$\begin{cases} f'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x + \dots \\ f'''(0) = 3 \cdot 2\, a_3 \end{cases} \Rightarrow a_3 = \frac{1}{3 \cdot 2} f'''(0).$$

...

In general one gets: $\boxed{a_n = \frac{1}{n!} f^{(n)}(0)}$

Thus:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$$

Some considerations are necessary:

- $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

  This is valid for each $x \in \mathbb{R}$ (which is the domain of the function).

- $f(x) = \frac{1}{1-x}$; $D = (-\infty, 1) \cup (1, \infty)$

  $f(x) = \sum_{n=0}^{\infty} x^n$ (i.e. $f^{(n)}(0) = n!$, therefore each coeff. $a_n = 1$ in this case!)

  However, this equivalence is valid only if $x = (-1, 1)$. This is namely the region of convergence of the series. (for $x = 2$: $\sum_{n=0}^{\infty} 2^n = \infty \dots$)

In general; $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!}$ is valid for $x \in D$, where D is the domain of $f(x)$ and x is part of D for which the summation is finite.

The precise definitions of "convergence" goes beyond the present discussion, but the intuitive meaning should be clear.

The Taylor-expansion is a good way to approximate a certain function $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{N} a_n x^n + O_q(x^{n+1}) \simeq \sum_{n=0}^{N} a_n x^n .$$

That is, in the vicinity of $x_0 = 0$ we can approximate the function $f(x)$ up to a given order.

<u>Example.</u>

$f(x) = \sin x$

$f(0) = \sin(0) = 0 = a_0;$

$f'(x) = +\cos x; \quad f'(0) = 1 = a_1;$

$f''(x) = -\sin x; \quad f''(0) = 0 = a_2;$

$f'''(x) = -\cos x; \quad f'''(0) = -1; \rightarrow a_3 = -\frac{1}{3!}$

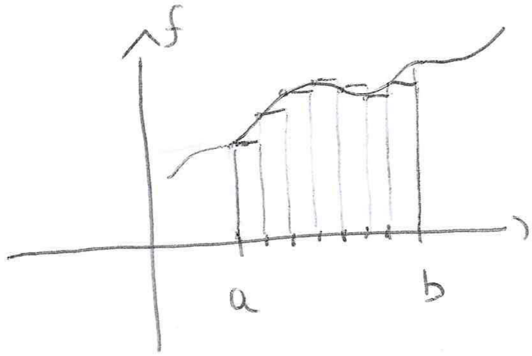
$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

(In this case $X = \mathbb{R}$).

When approximating $\sin x$ with $f(x, N) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ the approximation gets better and better - also for large x - when taking N large enough.

Integrals

$f(x) : D \subseteq \mathbb{R} \mapsto \mathbb{R}$ (a,b) segment $\subseteq D$.



Let us "divide" the segment in N small pieces of length δx / $N\delta x = b-a$.

(For simplicity let us take them all with equal length δx).

Let us also consider the points

$x_n = a + \delta x \cdot n$ $n = 0, 1, \dots N$

We then construct the sum

$$\sum_{n=0}^{N=(b-a)/\delta x} f(x_n)\,\delta x$$

The definite integral $\int_a^b f(x)\,dx$ is defined as the following limit:

$$\int_a^b f(x)\,dx = \lim_{\delta x \mapsto 0^+} \sum_{n=0}^{N=(b-a)/\delta x} f(x)\,\delta x$$

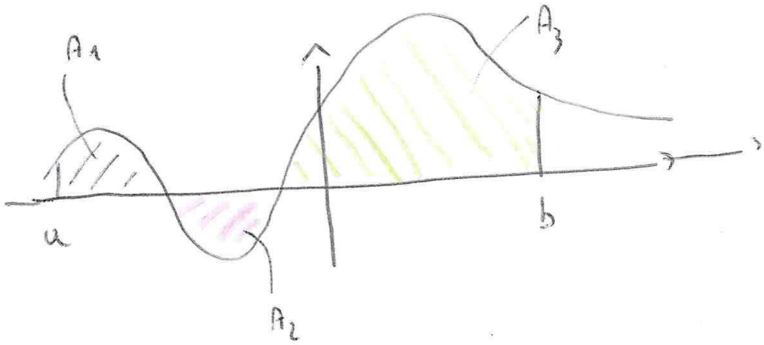
Intuitively: $\delta x \mapsto dx$ "infinitesimal"

$\sum_{n=0}^{N} \mapsto \int_a^b$ ($\int$ = long S, stands for sum (Latin)) it is clear that if $f(x) > 0$

$$I = \int_a^b f(x)\,dx$$

is the area between the x-axis and the function $f(x)$.

However, in general the sign is not always positive:



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 \;;$$

Out of the definition it follows that:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

and

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f\,dx + \beta \int_a^b g(x)\,dx.$$

Moreover, one defines that

$$\int_a^b f(x)\,dx = - \int_b^a f(x)\,dx$$

In this way the integral is well defined regardless if the lower limit is effectively smaller than the upper limit.