Particle with mass m in a potential $V(\vec{r})$

Wave function: $\psi(t,\vec{r}) : \mathbb{R} \times \mathbb{R}^3 \longmapsto \mathbb{C}$

The time-dependent Schrödinger equation is:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\vec{r})\, \psi$$

⇝ Schrödinger thought to describe a wave of matter, but it is not like that (e.g. double-slit experiment)

⇝ Born introduced the concept of "probability":

$|\psi(t,\vec{r})|^2 d^3r$ is the infinitesimal probability of finding at the instant t the particle in the infinit. volume d^3r.

Obviously, this prob. interpretation makes sense only if

$$\int |\psi(t,\vec{r})|^2 d^3r = 1 \quad \forall t$$

(The particle must be somewhere... integrated over all space, the prob. must be 1)

This is indeed the case, as the continuity eq shows.

Important properties of the Schr.-eq:

1) "Superposition principle" alias "linearity" of the Schr.-eq:

If ψ_1 and ψ_2 are two solutions of the Schr.-eq, then also the linear combination is such:

$$\psi = \alpha \psi_1 + \beta \psi_2$$

2) Continuity equation: out of the Schr.-eq it follows that

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{where} \quad \begin{cases} \rho = |\psi|^2 \\ \vec{J} = \frac{-i\hbar}{2m}\left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*\right) \end{cases}$$

$Q = \int d^3x \, \rho$ is such that

$$\frac{dQ}{dt} = \int d^3x \frac{\partial \rho}{\partial t} = \int d^3x \left(-\vec{\nabla} \vec{J}\right) = -\int_{S_\Omega} d\vec{S} \cdot \vec{J} = 0$$

$Q = \text{const}$!

Ergo: if $Q = 1$ at a certain "t", it is such $\forall t$.

3) The Schr.-eq is "deterministic"!

Given $\psi(0, \vec{x}) = f(\vec{x})$ (initial condition), $\psi(t, \vec{x})$ is perfectly defined $\forall t$ (both in the past and in the future).

It is the act of measurement (as for instance the measurement of position) which is not deterministic and causes a "sudden" and irreversible change (collapse) of the w.f.

Plane wave: $V(\vec{r})=0$.

$$\psi(t,\vec{r}) = N e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}$$

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi = -\frac{\hbar^2}{2m}\Delta\psi = \frac{\vec{p}^{\,2}}{2m}$$

Ergo, $\psi(t,\vec{r})$ is a solution of the Schr-eq if $\underbrace{E=\frac{\vec{p}^{\,2}}{2m}}_{\text{Classical dispersion relation}}$

Notice that, if we introduce the operator $\hat{\vec{p}} = -i\hbar\vec{\nabla}$, we have that:

$\hat{\vec{p}}\,\psi = \vec{p}\,\psi$ $\rightsquigarrow$ Eigenvector ψ with eigenvalue $\vec{p}$

Similarly, it follows that:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$$

is the so-called Hamilton operator.

The Schr-eq reads simply $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$

Note: the ~~w~~f ψ above is actually non-physical.

$|\psi|^2 = |N|^2 = $ const, $\int|\psi|^2 d^3r = \infty$.

One could normalise it in a box of volume V

with $N = \frac{1}{\sqrt{V}}e^{i\varphi}$, see details in the SCRIPT.

In general, if $\psi(t,\vec{r})$ is the wf, one has that

$$\boxed{\langle \vec{r} \rangle = \int d^3r \, \vec{r} \, |\psi(t,\vec{r})|^2 = (\psi(t,\vec{r}), \vec{r}\,\psi(t,\vec{r}))}$$

is the mean position at the instant t.

Simillarly:

$$\langle \vec{p} \rangle = (\psi(t,\vec{r}), \hat{\vec{p}}\,\psi(t,\vec{r})) = \int d^3r \, \psi^* \hat{\vec{p}} \, \psi =$$

$$= \int d^3r \, \psi^* (-i\hbar \vec{\nabla} \psi)$$

In general one has an operator $\hat{A} \equiv A(\vec{r}, \vec{p})$

~> Theory of operators on L^2 functions.

~> Hermitian operators as observables: $\hat{A} = \hat{A}^+$, whereas $(\psi_1, \hat{A}^+\psi_2) = (\psi_2, \hat{A}\psi_1)^*$

~> $\langle \hat{A} \rangle = (\psi, A\psi) = \int d^3r \, \psi^* \hat{A} \psi$ is the mean value of A.

~> Eigenvectors and eigenvalues of an operator $\hat{A}$ ($\hat{A}$ observable <-> $\hat{A}$ hermitian → Eigenvalues real)

ψ_m, $\hat{A}\psi_m = \lambda_m \psi_m$

By a measur. of $\hat{A}$, the possible outcomes are the eigenvalues λ_i.

~> $\vec{L} = \vec{r} \wedge \vec{p}$ $\quad$ $\vec{L}^2$ and L_φ are important operators.

Operator $\hat{A}(\vec{r}, \vec{p})$ (indep. on t)

Eigenvectors $\varphi_i(\vec{r})$ / $\hat{A}\,\varphi_m(\vec{r}) = \lambda\,\varphi_m(\vec{r})$ $m = 0, 1, 2, \ldots$

$\{\varphi_m(\vec{r})\}$ form an ONC basis.

ON $(\varphi_n, \varphi_m) = \int \varphi_n^* \varphi_m \, d^3r = \delta_{nm}$

C Each $\varphi(\vec{r})$ can be expressed as $\varphi(\vec{r}) = \sum_m c_m \varphi_m(\vec{r})$.
(well behaved)

The normalisation of $\varphi(\vec{r})$ implies that:

$$1 = \int d^3r\, |\varphi|^2 = \sum_m |c_m|^2$$

If at $t=0$ the wf is given by $\varphi(\vec{r}) = \varphi_m(\vec{r})$ then

$\langle \hat{A} \rangle = (\varphi, \hat{A}\varphi) = \lambda_m$. $\to$ if you do a meas. of $\hat{A} \to$ you get λ_m.

Moreover:

$\langle \hat{A}^2 \rangle = (\varphi, \hat{A}^2 \varphi) = \lambda_m^2$.

The so-called variance

$\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} = 0$ in this case.

$\varphi = \varphi_m \to$ we know for sure that the outcome is λ_m with probability "1".

In general:

$$\psi(\vec{r}) = \sum_n c_n \psi_n(\vec{r})$$

$$\langle \hat{A} \rangle = \sum_n |c_n|^2 \lambda_n \ .$$

In the framework of the prob. interpretation; if we measure $\hat{A}$, we always get an eigenvalue λ_n.

We have to repeat the exp many times and find that:

the value λ_n occurs with probability $|c_n|^2$.

Note, in this general case

$$\langle \hat{A}^2 \rangle = \sum_n |c_n|^2 \lambda_n^2 \Rightarrow \Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} \geq 0$$

After the measurement of $\hat{A}$ (at $t=0^+$) the state of the system collapses:

$$\underset{t=0^-}{\psi = \sum_m c_m \psi_m} \implies \underset{t=0^+}{\psi_n} \quad \text{with prob } |c_n|^2$$

⤳ Same structure in Hilbert-Raum ... when you have two operators $\hat{A}$ and $\hat{B}$ only if you have $[\hat{A},\hat{B}]=0$ you can build a basis of common eigenfunctions.

If $[\hat{A},\hat{B}] \neq 0$ it is not possible $\Rightarrow$ basis of uncert. relation.

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle i[A,B] \rangle| \Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2} \text{ because } [x,p] = i\hbar$$

Wave packet:

$$\psi = \psi(t,x) = \int_{-\infty}^{\infty} c(p)\, e^{-\frac{i}{\hbar}(Et - px)}$$

$E = \hbar\omega$
$p = \hbar k$

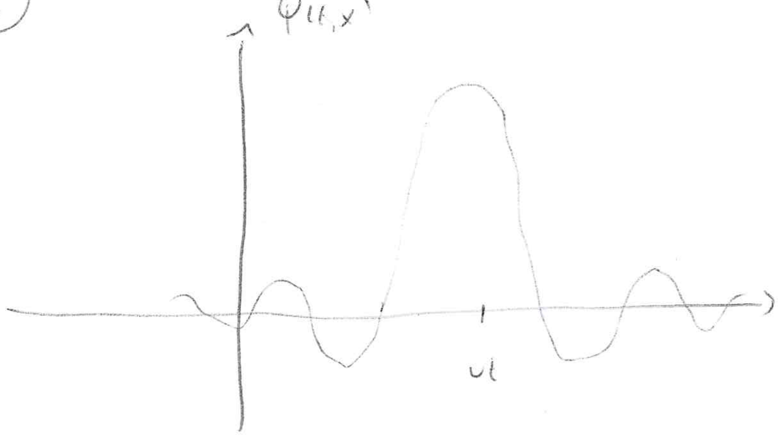
$$c(p) = \begin{cases} 1 & p \in (p_0 - \Delta p, p_0 + \Delta p) \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(t,x) = N \frac{\sin\left((x - vt)\frac{\Delta p}{\hbar}\right)}{x - vt} e^{-\frac{i}{\hbar}(E_0 t - p_0 x)}$$

with $E_0 = \frac{p_0^2}{2m}$

$v = \frac{p_0}{m}$ "velocity"

t



$\Delta x \cdot \Delta p = 2\pi\hbar$ in this case.

We have the analogous situation of a classical particle!

Stationary (Time-indep) Schr-eq:

$$\psi(t,\vec{r}) = f(t)\,\varphi(\vec{r})$$

$$i\hbar\,\varphi(\vec{r})\frac{\partial f(t)}{\partial t} = -\frac{\hbar^2}{2m} f(t)\,\Delta\varphi(\vec{r}) + f(t)V(\vec{r})\varphi(\vec{r})$$

We look for a sol of the form

$$f(t) = e^{-\frac{i}{\hbar}Et}$$

$$\Downarrow$$

$$E\,\varphi(\vec{r})\,\cancel{f(t)} = \cancel{f(t)}\,H\varphi(\vec{r})$$

$$\boxed{H\varphi(\vec{r}) = E\varphi(\vec{r})} \to \text{Spectrum of the hamiltonian.}$$

Now, when we solve it we already now that

$\varphi(\vec{r})e^{-\frac{i}{\hbar}Et}$ is the "full sol." of the time-dep Schr eq.,

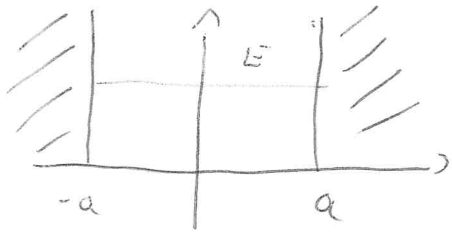
$$|\varphi(\vec{r})e^{-\frac{i}{\hbar}Et}|^2 = |\varphi(\vec{r})|^2 \text{ is indep. on } t!$$

1 dim case

$$H\,\psi(x) = E\,\psi(x)$$

- solvable
- simple
- visualization

$$V(x) = \begin{cases} \infty & |x| > a \\ 0 & |x| < a \end{cases}$$



$|x| < a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0} \Rightarrow \frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$\begin{cases} \psi = a e^{ikx} + b e^{-ikx} \equiv A\sin(kx) + B\cos(kx) & |x| < a \\ \psi = 0 & |x| > a \end{cases}$$

Continuity of $\psi(\pm a) = 0$

$A\sin(ka) = 0$, $B\cos(ka) = 0$

$$\begin{cases} \psi = B\cos(k_{2n+1}x) & k_{2n+1} = (2n+1)\frac{\pi}{2a} & n = 0, 1, \dots \\ \psi = A\sin(k_{2n}x) & k_{2n} = 2n\frac{\pi}{2a} & n = 1, 2, \dots \end{cases}$$

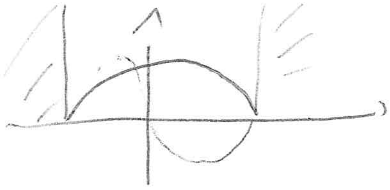
The energy levels

$$E_n = \frac{\hbar^2 \pi^2}{8ma^2} n^2 \qquad n = 1, 2, \ldots$$

Recall:

$$E = \frac{p^2}{2m}$$

De-Broglie: $\lambda = \frac{h}{p}$



$\lambda_1 = 4a$ $\qquad \lambda_n = \frac{4a}{n} \qquad n = 1, 2, \ldots$

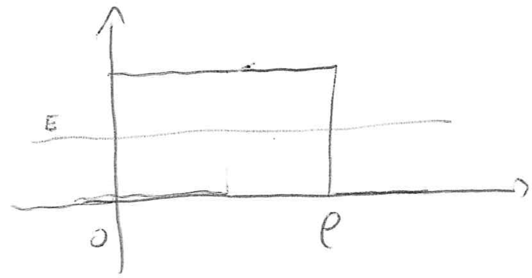
$\lambda_2 = 2a$

$\ldots$

Ergo: $p_n = \frac{h}{\lambda_n} = \frac{h}{4a} \cdot n$

$$E = E_n = \frac{p_n^2}{2m} = \frac{h^2 n^2}{2 \cdot 16 a^2 m} \cdot \frac{4\pi^2}{4\pi^2} = \frac{\hbar^2 \pi^2}{8ma^2} n^2 \ldots$$

$E_1 > 0 \rightarrow$ there must be motion, in agreement with the uncertainty principle.



$x \in (0, \ell)$

$E < V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

(The full problem is indeed complicated...
One has to put the $\neq$ functions together by imposing the continuity of $\psi(x)$ and $\psi'(x)$ in $x=0$ and $x=\ell$)

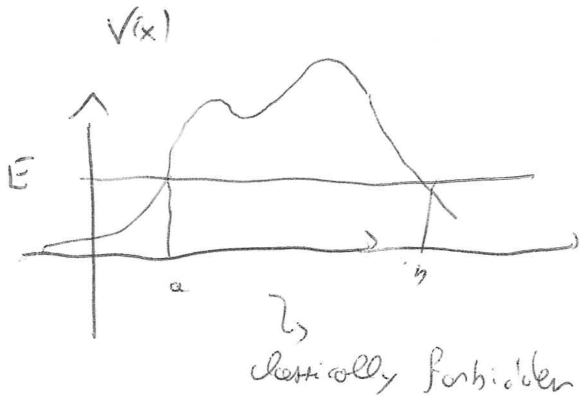
$$\frac{d^2\psi}{dx^2} - \frac{2m(V_0-E)}{\hbar^2}\psi = 0$$

$$\psi(x) = Ae^{\beta x} + Be^{-\beta x} \qquad \beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$T \simeq \left|\frac{\psi(\ell)}{\psi(0)}\right|^2 = e^{-2\beta\cdot\ell}$$

$$E = \frac{p^2}{2m} + V(x) \geq V(x)$$

For a general form

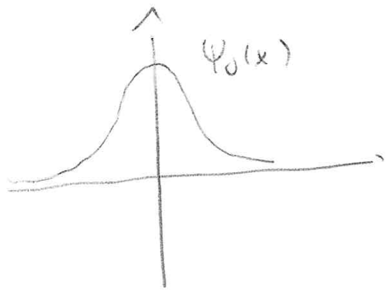


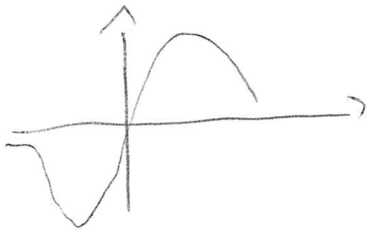
$$T \simeq e^{-2\int_a^b \sqrt{\frac{2m}{\hbar^2}(V(x)-E)}}$$

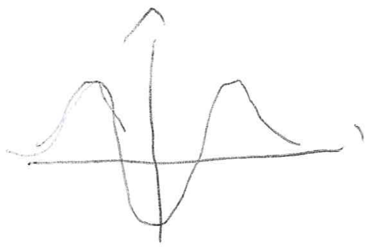
$$V(x) = \frac{1}{2} m \omega^2 x^2$$

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega \qquad n = 0, 1, 2, ..$$

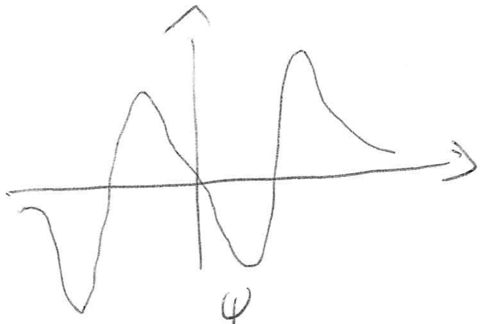
$$\psi_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}} \; e^{-\frac{1}{2}\alpha^2 x^2} \qquad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$



$\psi_1(x) =$  $\approx (\alpha x)\, e^{-\frac{1}{2}\alpha x^2}$



ψ_3



ψ_4

Central case

$V(r) \qquad r = |\vec{r}|$

$$\psi(\vec{r}) = \varphi_\ell(r)\, \underbrace{Y_{\ell m}(\vartheta, \varphi)}$$

There are known functions which diagonalise $\vec{L}^2$ and L_z:

$$\vec{L}^2\, Y_{\ell m}(\vartheta,\varphi) = \hbar^2 \ell(\ell+1)\, Y_{\ell m}(\vartheta,\varphi)$$

$$L_z\, Y_{\ell m}(\vartheta,\varphi) = \hbar m\, Y_{\ell m}(\vartheta,\varphi)$$

$\rightsquigarrow$ after performing the calculation and setting

$$u_\ell = r\varphi_\ell$$

we get:

$$-\frac{\hbar^2}{2m}\frac{d^2 u_\ell}{dr^2} + \left[V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right] u_\ell = E u_\ell$$

Normal 1-dim Schr.-eq with $\boxed{V_{eff}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}}$

One can show that:

$$\int_0^\infty |u_\ell(r)|^2\, dr = 1$$

$|u_\ell(r)|^2 dr$ is the prob. to find the particle between r and $r+dr$!

$$V(r) = -\frac{Ze^2}{r}$$

$$E = -\frac{E_0}{(n'+\ell+1)^2} \qquad E_0 \simeq \frac{Z^2 e^2}{2a_0} \qquad a_0 = \frac{\hbar}{me^2}$$

$$n = n'+\ell+1$$

$$E_n = -\frac{E_0}{n^2} \qquad n = 1, 2, 3, \ldots$$

$$\ell = 0, \ldots n-1$$

$$m = -\ell, \ldots, \ell$$

Hilbert-Raum $\{|a\rangle\}$

$\{|n\rangle\}$ $n=1,2,\dots$ ONC basis.

Scalar product $\langle a|b\rangle$ is def.

$|a\rangle$ state "KET"

$\langle a|$ "operator" (Bra)

ON $\langle n|m\rangle = \delta_{nm}$

C $\sum_n |n\rangle\langle n| = 1$

Consider a physical state $|s\rangle$

$|s\rangle = \sum_n c_n |n\rangle$ $\qquad \langle n|s\rangle = c_n$

Basis of positions

$\langle\vec{r}|s\rangle = \underbrace{\psi(\vec{r})}_{\text{wf in position space}}$

$\{|\vec{r}\rangle\}$ is a basis $\langle\vec{r}|\vec{r}\,'\rangle = \delta(\vec{r}-\vec{r}\,')$

$|s\rangle = \int d^3r\, \psi(\vec{r})\,|\vec{r}\rangle$

$\langle\vec{r}\,'|s\rangle = \int d^3r\, \psi(\vec{r})\,\delta(\vec{r}\,'-\vec{r}) = \psi(\vec{r}\,')$ qed

$|s(t)\rangle$

Time-dep Schr eq is simply

$$i\hbar \frac{\partial}{\partial t}|s(t)\rangle = H|s(t)\rangle$$

$\longrightarrow$

$$|s(t)\rangle = U(t)|s(0)\rangle = e^{-\frac{i}{\hbar}Ht}|s(0)\rangle$$ H indep. on t

$$i\hbar\frac{\partial|s(t)\rangle}{\partial t} = H|s(t)\rangle$$ q.e.d

$$U(t) = e^{-\frac{i}{\hbar}Ht}$$ time evolution operator.

–

operator $\hat{A}$

$$\hat{A}|m\rangle = \lambda_m|m\rangle \qquad m = 1, 2, \ldots$$

$$|s\rangle = \sum_m c_m|m\rangle \qquad \sum_m |c_m|^2 = 1$$ just as discussed before.

–

Let us now consider

$|s_1(t)\rangle$, $|s_2(t)\rangle$ and the operator $\hat{A}(t)$

eventual "explicit" dep. on t

$$\langle s_2(t)|\hat{A}(t)|s_1(t)\rangle = \langle s_2(0)|\underbrace{e^{\frac{i}{\hbar}Ht}\hat{A}(t)e^{-\frac{i}{\hbar}Ht}}|s_1(0)\rangle$$

$$\hat{A}_H(t) = e^{\frac{i}{\hbar}Ht}\hat{A}(t)\,e^{-\frac{i}{\hbar}Ht}$$

in Schr.: the operator do no dep. on time (unless they have some exp. dependence)

In Heisenberg: the states do not depend on time

Full description:

$${}_S\langle S_2 | \hat{A}_S | S_1 \rangle_S = {}_H\langle S_2 | \hat{A}_H | S_1 \rangle_H \quad \text{by construction.}$$

$$\hat{A}_H = e^{\frac{i}{\hbar} H t} \hat{A}_S e^{-\frac{i}{\hbar} H t}$$

$$\frac{d\hat{A}_H}{dt} = \frac{i}{\hbar} H \hat{A}_H - \frac{i}{\hbar} \hat{A}_H H = \frac{i}{\hbar} [H, \hat{A}_H]$$

Note:

$$\hat{H}_H = \hat{H}_S$$

[if $\hat{A}_H$ has also an explicit dep. on time: $\frac{d\hat{A}_H}{dt} = \frac{i}{\hbar}[H, \hat{A}_H] + \frac{\partial \hat{A}_H}{\partial t}$.]

$\hat{A}$ is linear if

$$\hat{A}(\alpha|a\rangle + \beta|b\rangle) = \alpha\hat{A}|a\rangle + \beta\hat{A}|b\rangle$$

Given $\hat{A}$, one defines the operator $\hat{A}^+$ /

$$\langle a|\hat{A}^+|b\rangle = \langle b|\hat{A}|a\rangle^* \quad \forall\ |a\rangle, |b\rangle \in \mathcal{H}$$

[This means: $\{|m\rangle\}$

$$\langle m|\hat{A}|n\rangle = A_{mn}$$

$$\langle n|\hat{A}^+|m\rangle = \langle m|A|n\rangle^* = A^*_{mn}$$

Matrix A

$\Downarrow$

Matrix $A^+ = A^{t*}$]

Consider now a basis of eigenvectors

$$\hat{A}|m\rangle = \lambda_m|m\rangle$$

$$\langle n|\hat{A}|m\rangle = \lambda_m\delta_{nm} = A_{nm}$$

$$\langle m|\hat{A}^+|n\rangle = \langle n|\hat{A}|m\rangle^* = A^*_{nm} = \lambda^*_m\,\delta_{nm}$$

Ergo we see that $\hat{A} = \hat{A}^+ \longleftrightarrow$ real eigenvalues!

Only hermitian operators can represent physical observables

(The question if this is true also viceversa is still debated)

Beispiel:

$\{|1\rangle, |2\rangle, |3\rangle\}$ $\quad H|n\rangle = E_n|n\rangle$

is the state

$|S\rangle = |1\rangle + |2\rangle + |3\rangle$

physical?

Answer: No! $\langle S|S\rangle = 3 \neq 1$

In order to make it physical:

$$|S\rangle = \frac{e^{i\phi_1}}{\sqrt{3}}|1\rangle + \frac{e^{i\phi_2}}{\sqrt{3}}|2\rangle + \frac{e^{i\phi_3}}{\sqrt{3}}|3\rangle = \sum_n c_n|n\rangle \qquad c_n = \frac{e^{i\phi_n}}{\sqrt{3}}$$

$|c_n|^2 = \frac{1}{3}$

consider the operator H; if we make a meas of the energy:

Note: $\langle 1|S\rangle = \frac{e^{i\phi_1}}{\sqrt{3}} \rightarrow$

$\frac{1}{3} \rightarrow E_1, \quad \frac{1}{3} \rightarrow E_2, \quad \frac{1}{3} \rightarrow E_3.$ $\qquad P_1 = |\langle 1|S\rangle|^2 = \frac{1}{3}.$

Now, let us consider a following question:

which is the probability to find the state as $|A\rangle = \sqrt{\frac{1}{2}}(|1\rangle + |2\rangle)$?

$$\langle A|S\rangle = \sqrt{\frac{1}{2}}(\langle 1| + \langle 2|)|S\rangle = \sqrt{\frac{1}{2}}\frac{e^{i\phi_1}}{\sqrt{3}} + \sqrt{\frac{1}{2}}\frac{e^{i\phi_2}}{\sqrt{3}}$$

$$= \sqrt{\frac{1}{6}}(e^{i\phi_1} + e^{i\phi_2})$$

Ergo the prob. is $\frac{1}{6}|e^{i\phi_1} + e^{i\phi_2}|^2 = \frac{1}{6}(e^{i\phi_1} + e^{i\phi_2})(e^{-i\phi_1} + e^{-i\phi_2}) =$

$$= \frac{1}{6}(2 + e^{i\Delta\phi} + e^{-i\Delta\phi}) = \frac{1}{6}(2 + 2\cos(\Delta\phi))$$

NB:

$O_A = |A\rangle\langle A|$

$\langle S | O_A | S \rangle = \langle S | A \rangle \langle A | S \rangle = |\langle A | S \rangle|^2$ is the prob.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$H|n\rangle = E_n |n\rangle \qquad n = 0, 1, 2, \ldots$$

$$a = \frac{1}{\sqrt{2m\hbar\omega}} (p - im\omega x)$$

$$[x, p] = i\hbar \rightarrow [a, a^\dagger] = 1 \,, \qquad [a, a] = [a^\dagger, a^\dagger] = 0 .$$

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

$|0\rangle \equiv$ state with "minimal energy"

Consider:
$a|n\rangle$ has energy $(E_n - \hbar\omega)$

Ergo:

$$a|0\rangle = 0 \rightarrow E_0 = \frac{\hbar\omega}{2}$$

$$a^\dagger |0\rangle$$

$$\hbar\omega \left(a^\dagger a + \frac{1}{2}\right) a^\dagger |0\rangle = \underbrace{\hbar\omega \left(1 + \frac{1}{2}\right)}\, a^\dagger |0\rangle$$

$$E_1 = \frac{3}{2} \hbar\omega$$

...

$$(a^\dagger)^n |0\rangle \rightarrow E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

More in detail

$\langle x|0\rangle = N e^{-\frac{1}{2}\alpha x^2}$

$H = \hbar\omega(a^\dagger a + \frac{1}{2})$

$[H,a] = \hbar\omega[a^\dagger a, a] = -\hbar\omega$

$[H,a^\dagger] = \hbar\omega$

$H|n\rangle = E_n|n\rangle$

$H a|n\rangle = (aH - \hbar\omega a)|n\rangle = (a E_n - \hbar\omega a)|n\rangle = (E_n - \hbar\omega)a|n\rangle$!

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \quad \rightarrow \quad i\hbar \frac{\partial \psi}{\partial t}\psi^* = (H\psi)\psi^*$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = H\psi^* \qquad -i\hbar \frac{\partial \psi^*}{\partial t}\psi = (H\psi^*)\psi$$

$$i\hbar \frac{\partial |\psi|^2}{\partial t} = (H\psi)\psi^* - (H\psi^*)\psi$$

$$= -\frac{\hbar^2}{2m}\left[(\Delta\psi)\psi^* - (\Delta\psi^*)\psi\right]$$

$$= -\frac{\hbar^2}{2m}\vec{\nabla}\cdot\left[(\vec{\nabla}\psi)\psi^* - (\vec{\nabla}\psi^*)\psi\right]$$

$=$