TIME EVOLUTION IN HILBERT SPACE

$\psi(0,\vec{x}) = f(\vec{x}) \mapsto |f\rangle \quad ; \quad f(\vec{x}) = \langle \vec{x} | f \rangle$

$\psi(t,\vec{x}) \quad \longmapsto |\psi(t)\rangle \quad ; \quad \psi(t,\vec{x}) = \langle \vec{x} | \psi(t) \rangle$

$|\psi(t)\rangle$ is the time-dep. ket.

$\psi(t,\vec{x})$ solves the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H_{\vec{x}} \psi \qquad H_{\vec{x}} = -\frac{\hbar^2}{2m} \Delta + V$$

But then:

$$\boxed{i \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle} \qquad \langle \vec{x} | H | \vec{x}' \rangle = H_{\vec{x}} \, \delta(\vec{x} - \vec{x}')$$

Namely by multiplying by $\langle \vec{x} |$ from the left

$$i \frac{\partial}{\partial t} \psi(t,\vec{x}) = \langle \vec{x} | H | \psi(t) \rangle = \int d^3x' \, \langle \vec{x} | H | \vec{x}' \rangle \langle \vec{x}' | \psi(t) \rangle =$$

$$= \int d^3x' \, H_{\vec{x}} \, \delta(\vec{x} - \vec{x}') \, \psi(t, \vec{x}')$$

$$= H_{\vec{x}} \, \psi(t, \vec{x})$$

$$\text{So } i \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle$$

is the Schrödinger eq. using the formalism of the Hilbert space.

FORMAL SOLUTION OF THE SCHR. EQ:

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle$$

$$|\psi(0)\rangle = |\psi_0\rangle \equiv |f\rangle$$

Ergo:

$$|\psi(t)\rangle = \underbrace{e^{-iHt}}_{\text{operator}} |\psi_0\rangle$$ solves the Schr. eq.

Plug in and you can verify that...

$U = e^{-iHt}$ is the time-evolution operator.
It "takes" the state $|\psi_0\rangle$ and delivers $|\psi(t)\rangle$.

$U = e^{-iHt}$ is crucial in QM (and in QFT as well...).

$U \equiv U(0,t)$

$$\left[U(t_1, t_2) = e^{-iH(t_2 - t_1)} \quad / \quad U(t_1, t_2)|\psi(t_1)\rangle = |\psi(t_2)\rangle \right]$$

The Schrödinger time-indep. eq. is obtained by setting

$$|\psi(t)\rangle = e^{-iEt}|\psi_0\rangle$$

You immediately get $\quad H|\psi_0\rangle = E|\psi_0\rangle$

Eigenvalue problem in the Hilbert space

⇝ example of the 'Harmonic oscillator'

Let us consider for simplicity a 1D problem.

At $t = t_i$ the particle is in the initial point $|x_i\rangle$

$|\psi(0)\rangle = |\psi_i\rangle = |x_i\rangle$ $\qquad\qquad$ $(\psi_i(x) = \psi(t_i, x) = \delta(x - x_i))$

We thus know where the particle starts...

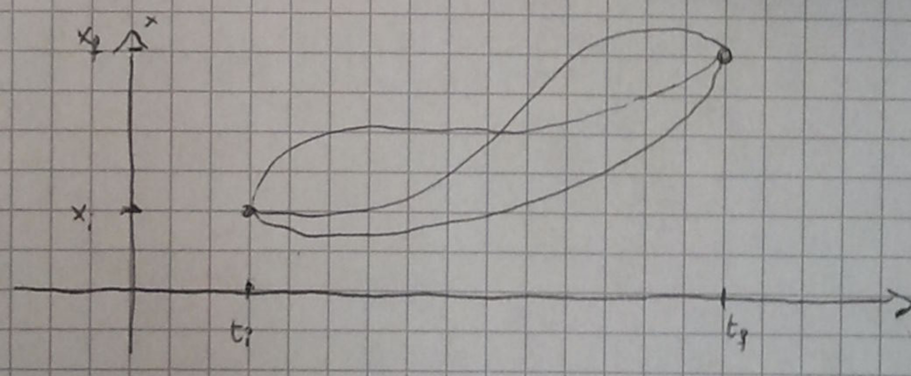
The quantum evolution implies that at $t > t_i$:

$$\boxed{|\psi(t)\rangle = e^{-iH(t-t_i)} |x_i\rangle.}$$ $\qquad$ x_i = initial.

Now, suppose that after the time T we make a meas. of position: in particular, we measure if the particle is in the point x_f

x_f = final

$A(x_i \mapsto x_f) = \langle x_f | \psi(t)\rangle = \langle x_f | e^{-iHT} | x_i\rangle$. $\qquad$ $T = t_f - t_i$



Feynman shows us that:

$$\langle x_f | e^{-\frac{H}{\hbar}(t_f - t_i)} | x_i\rangle = \int_{\substack{x(t_i) = x_i \\ x(t_f) = x_f}} Dx(t)\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(x_i, \dot{x}_i)\, dt} = \int_{BC} Dx(t)\, e^{iS[x(t)]}.$$

All paths contribute!!

(and actually they do that with the same amplitude...) How is it possible?

How does classical mechanics emerge?