Operators

An operator in $\mathcal{F}$ is such that: $\hat{A}: \mathcal{F} \mapsto \mathcal{F}$

($|f_2\rangle\langle f_1|$ is an example of operator).

Ergo, if $|f\rangle \in \mathcal{F}$ one has that:

$\hat{A}|f\rangle \in \mathcal{F}$

We are interested in linear operators:

$$\hat{A}(\alpha_1 |f_1\rangle + \alpha_2 |f_2\rangle) = \alpha_1 \hat{A}|f_1\rangle + \alpha_2 \hat{A}|f_2\rangle$$

In $\hat{L}^2$ we had operators of the type $\hat{A}_{\vec{x}}: L^2 \mapsto L^2$. Here it is the very same thing...

For instance, $\hat{P}_x \equiv -i\frac{\partial}{\partial x}$ is such: $-i\frac{\partial}{\partial x}$

$$\underset{\in L^2}{f(\vec{x})} \mapsto \underset{\in L^2}{A_{\vec{x}} f(\vec{x})}$$

THE CONNECTION IS:

$$\underset{L^2 \mapsto L^2}{\hat{A}_{\vec{x}}} \longleftrightarrow \underset{\mathcal{F} \mapsto \mathcal{F}}{\hat{A}} \quad \text{if} \quad \hat{A}|f\rangle = |\hat{A}_{\vec{x}} f\rangle \quad \forall \quad |f\rangle \in L^2$$

Explicitely:

$$\begin{cases} f_1(\vec{x}) \mapsto \hat{A}_{\vec{x}} f_1(\vec{x}) = f_2(\vec{x}) \\ |f_1\rangle \mapsto \hat{A}|f_1\rangle = |f_2\rangle \end{cases}$$

$\langle \vec{x}|f_1\rangle = f_1(\vec{x}) \qquad \langle \vec{x}|A|f_1\rangle = \langle \vec{x}|f_2\rangle = f_2(\vec{x})$.

It then follows that:

$$\langle \vec{x} | \hat{A} | f_1 \rangle = f_2(\vec{x}) = \int d^3x' \, \langle \vec{x} | \hat{A} | \vec{x}' \rangle \langle \vec{x}' | f_1 \rangle$$

$$= \int d^3x' \, \langle \vec{x} | \hat{A} | \vec{x}' \rangle \, f_1(\vec{x}')$$

$$\langle \vec{x} | \hat{A} | \vec{x}' \rangle = \hat{A}_{\vec{x}} \, \delta(\vec{x} - \vec{x}') = \delta(\vec{x} - \vec{x}') \, A_{\vec{x}'}$$

$$\langle \vec{x} | \hat{A} | f_1 \rangle = \int d^3x' \, A_{\vec{x}} \, \delta(\vec{x} - \vec{x}') \, f_1(\vec{x}') = A_{\vec{x}} \int d^3x' \, \delta(\vec{x} - \vec{x}') \, f_1(\vec{x}') = A_{\vec{x}} \, f_1(\vec{x}).$$

Equivalently:

$$\langle \vec{x} | \hat{A} | f_1 \rangle = \int d^3x' \, \delta(\vec{x} - \vec{x}') \, A_{\vec{x}'} \, f_1(\vec{x}') = A_{\vec{x}} \, f_1(\vec{x}).$$

subtlety, that is quite important in a correct mathematical framework.

CONSIDER THE OPERATOR $\hat{A}: \mathcal{F} \mapsto \mathcal{F}$.

THE HERMITIAN CONJ. $\hat{A}^{\dagger}$ IS DEFINDED AS:

$$\langle f_1|\hat{A}^{\dagger}|f_2\rangle = \langle f_2|\hat{A}|f_1\rangle^{*}$$

- IF WE CONSIDER A DISCRETE BASIS $\{|\varphi_n\rangle\}$ AN OPERATOR $\hat{A}$ IS EQUIVALENTLY DESCRIBED BY THE MATRIX

$$\langle \varphi_n|\hat{A}|\varphi_m\rangle = A_{nm} \qquad A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

$\hat{A}^{\dagger}$ is such that

$$\langle \varphi_n|\hat{A}^{\dagger}|\varphi_m\rangle = \langle \varphi_m|\hat{A}|\varphi_n\rangle^{*} \equiv \begin{pmatrix} A_{11}^{*} & A_{21}^{*} \\ A_{12}^{*} & A_{22}^{*} \end{pmatrix} = A^{t*}$$

- AN OPERATOR IS HERMITIAN IF $\hat{A} = \hat{A}^{\dagger}$.

Property: $\hat{A}$ hermitian $\mapsto \lambda_n$ real.

Consider now a special basis $\{|\varphi_n\rangle\}$ which is made of eigenstates of $\hat{A}$:

$$\hat{A}|\varphi_n\rangle = \lambda_n|\varphi_n\rangle$$

$$\lambda_n = \langle \varphi_n|\hat{A}|\varphi_n\rangle =$$

$$\underset{\hat{A}=\hat{A}^{\dagger}}{=} \langle \varphi_n|\hat{A}^{\dagger}|\varphi_n\rangle \underset{\text{def}}{=} \langle \varphi_n|\hat{A}|\varphi_n\rangle^{*} = \lambda_n^{*} \mapsto \lambda_n = \lambda_n^{*} \mapsto \lambda_n \text{ real}.$$

The inverse is also true.

Observables $\mapsto$ Hermitian operators (inverse: ?).

DIGRESSION; $|f\rangle^+$ and $(\hat{A}|f\rangle)^+$.

Consider

$$\hat{A}|f_1\rangle = |f_2\rangle.$$

Then:

$$\langle f_2| = \langle f_1|\hat{A}^+$$

FIRST, IT IS CUSTOM TO WRITE $\boxed{|f\rangle^+ = \langle f|}$ ACTUALLY CARE IS NEEDED) BECAUSE $|f\rangle$ IS A STATE AND STRICTLY SPEAKING NOT AN OPERATOR.

LET US CONSIDER $\hat{O} : \mathcal{F} \mapsto \mathbb{C}$ $\quad \hat{O}|\varphi\rangle$ is a complex number.

$\hat{O}^+ : \mathcal{F} \mapsto \mathbb{C}$ $\quad \hat{O}^+|\varphi\rangle = (\hat{O}|\varphi\rangle)^*$

This is the definition of $\hat{O}^+$ for an operator $\mathcal{F} \mapsto \mathcal{F}$.

Let us consider $\hat{O}_{|f\rangle}$ such that $\hat{O}_{|f\rangle}|\varphi\rangle = \langle\varphi|f\rangle$

Then, we realize that $\hat{O}^+_{|f\rangle} = \langle f|$.

Note, $\hat{O}_f$ is basically eq. to $|f\rangle$. $|f\rangle$ can be seen as an operator which associates $|\varphi\rangle$ to $\langle\varphi|f\rangle$.

Unitary operator: consider $\hat{U}: \mathcal{F} \mapsto \mathcal{F}$.

$\hat{U}$ is unitary if $\hat{U}^{\dagger} = \hat{U}^{-1}$.

UNITARY OPERATORS ARE EXTREMELY IMPORTANT IN PHYSICS.

THE REASON: THEY PRESERVE THE NORM.

$|f_1\rangle \;/\; \langle f_1|f_1\rangle = 1$.

$|f_2\rangle = U|f_1\rangle \mapsto \langle f_2|f_2\rangle = \langle f_1|U^{\dagger}U|f_1\rangle = \langle f_1|f_1\rangle = 1$.

A unitary operator can be expressed as:

$$\hat{U} = e^{i\hat{A}} = 1 + i\hat{A} + \frac{1}{2}(i\hat{A})^2 + \cdots$$

where $\hat{A}$ is Hermitian.

Namely, for Hermitian $\hat{A}$:

$$\hat{U}^{\dagger} = (e^{i\hat{A}})^{\dagger} = \left(1 + i\hat{A} + \frac{1}{2}(i\hat{A})^2 + \cdots\right)^{\dagger} = \left(1 - i\hat{A} + \frac{1}{2}(-i\hat{A})^2 \cdots\right) = e^{-i\hat{A}}.$$

Then:

$$\hat{U}^{\dagger}\hat{U} = e^{-i\hat{A}}e^{i\hat{A}} = e^{-i\hat{A}+i\hat{A}} = e^{\hat{0}} = \hat{1} \quad \text{qed.}$$

ACHTUNG: $e^{\hat{A}}e^{\hat{B}}$ IS IN GENERAL $\neq$ from $e^{\hat{A}+\hat{B}}$, but:

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}] + \underbrace{\cdots}_{\text{very complicated expression}}}$$