$\psi(0, \vec{x}) = f(\vec{x})$ (wave function at $t=0$ to be definite...). $f(\vec{x}) \in L^2(\mathbb{R}^3)$, which means that $\int_V |f(\vec{x})|^2 d^3x < \infty$.

Hilbert space $\mathcal{F}$ is introduced via the correspondence

$f(\vec{x}) \longleftrightarrow |f\rangle \in \mathcal{F}$ (ket $|f\rangle$). One has:

$(f_1, f_2) = \int d^3x\, f_1^*(\vec{x}) f_2(\vec{x})$ is a scalar product. $(\cdot,\cdot): L^2 \times L^2 \mapsto \mathbb{C}$.

$(f_1, f_2) = \langle f_1 | f_2 \rangle = \int d^3x\, f_1^*(\vec{x}) f_2(\vec{x})$. NOTE: $\langle f_2 | f_1 \rangle = \langle f_1 | f_2 \rangle^*$

Note, the "bra" $\langle f_1|$ can be seen as an operator: $L^2 \mapsto \mathbb{C}$.
Namely, $\langle f_1| (|f\rangle) = \langle f_1 | f \rangle$ associates $|f\rangle$ to the complex number $\langle f_1 | f \rangle$.

Now, in L^2 a basis is given for instance by the set of function $\{\varphi_n(\vec{x})\}$. The basis is ON if $(\varphi_n, \varphi_m) = \int d^3x\, \varphi_n^*(\vec{x}) \varphi_m(\vec{x}) = \delta_{nm}$ and is complete if each function $f(\vec{x}) \in L^2$ can be expressed as

$$f(\vec{x}) = \sum_n c_n \varphi_n(\vec{x})$$

The c_m can be obtained by multiplying by $\varphi_m^*(\vec{x})$ and integrating:

$$\varphi_m^*(\vec{x}) f(\vec{x}) = \sum_n c_n \varphi_m^*(\vec{x}) \varphi_n(\vec{x}) \mapsto \int d^3x\, \varphi_m^*(\vec{x}) f(\vec{x}) = (\varphi_m, f) = \sum_n c_n (\varphi_m, \varphi_n)$$
$$= \sum_n c_n \delta_{mn} = c_m$$

Ergo: $c_m = (\varphi_m, f)$

In the language of $\mathcal{F}$ the set of function $\{\varphi_n(\vec{x})\}$ is replaced by $\{|\varphi_n\rangle\}$ with $\langle \varphi_n | \varphi_m \rangle = \int d^3x\, \varphi_n^*(x) \varphi_m(x) = \delta_{nm}$.

Then, each ket $|f\rangle$ can be rewritten as

$$|f\rangle = \sum_n c_n |\varphi_n\rangle$$

In order to determine the c_m one act from the left with the operator $\langle\varphi_m|$:

$$\langle\varphi_m|f\rangle = \sum_n c_n \langle\varphi_m|\varphi_n\rangle = \sum_n c_n \delta_{mn} = c_m$$

So $c_m = \langle\varphi_m|f\rangle = \int d^3x\, \varphi_m^*(\bar{x}) f(\bar{x})$ (just as before).

In the present language one can easily prove that a ONC basis is such that:

$$\sum_n |\varphi_n\rangle\langle\varphi_n| \; 1$$

First, note that $\langle f_1| : \mathcal{F} \mapsto \mathbb{C}$; However, $|f_2\rangle\langle f_1| : \mathcal{F} \mapsto \mathcal{F}$.

Namely:

$$\underbrace{|f_2\rangle\langle f_1| : |f\rangle}_{\in \mathcal{F}} \longmapsto \underbrace{\langle f_1|f\rangle |f_2\rangle}_{\in \mathcal{F}}$$

So indeed $\sum_n |\varphi_n\rangle\langle\varphi_n| : \mathcal{F} \mapsto \mathcal{F}$. But:

$$\underbrace{\left(\sum_n |\varphi_n\rangle\langle\varphi_n|\right)}_{1_{\mathcal{F}} = 1} |f\rangle = \sum_n \underbrace{\langle\varphi_n|f\rangle}_{c_n} |\varphi_n\rangle = \sum_n c_n |\varphi_n\rangle = |f\rangle$$

One writes simply: $\sum_n |\varphi_n\rangle\langle\varphi_n| = 1 \rightarrow$ completeness.

EXAMPLE:

$\{|1\rangle, |2\rangle, |3\rangle\}$ $\quad H|n\rangle = E_n|n\rangle$

- IS THE STATE $|f\rangle = |1\rangle + |2\rangle + |3\rangle$ physical $\rightarrow$ NO BECAUSE $\langle f|f\rangle = 3$.

- A PHYSICAL STATE IS FOR INSTANCE

$$|f\rangle = \frac{e^{i\phi_1}}{\sqrt{3}}|1\rangle + \frac{e^{i\phi_2}}{\sqrt{3}}|2\rangle + \frac{e^{i\phi_3}}{\sqrt{3}}|3\rangle = \sum_n c_n|n\rangle$$

$$c_n = \frac{e^{i\phi_n}}{\sqrt{3}} \qquad |c_n|^2 = \frac{1}{3}$$

If we make a meas. of H we find E_1 with probability $1/3$, E_2 with prob. $1/3$ and E_3 with prob. $1/3$.

$$P_{E_1} = |\langle 1|f\rangle|^2 \quad \ldots$$

- WHICH IS THE PROB. THAT THE STATE IS DESCRIBED BY $|A\rangle = \sqrt{\frac{1}{2}}(|1\rangle + |2\rangle)$?

$$P = |\langle A|f\rangle|^2 = \left|\sqrt{\frac{1}{2}}(\langle 1| + \langle 2|)\left(\frac{e^{i\phi_1}}{\sqrt{3}}|1\rangle + \frac{e^{i\phi_2}}{\sqrt{3}}|2\rangle + \frac{e^{i\phi_3}}{\sqrt{3}}|3\rangle\right)\right|^2$$

$$= \left|\sqrt{\frac{1}{6}}\left(e^{i\phi_1} + e^{i\phi_2}\right)\right|^2 = \frac{1}{3}(1 + \cos(\Delta\phi))$$

AQM - Recall of Hilbert space $\{|\vec{x}\rangle, |\vec{p}\rangle\}$

A BASIS IN 3D IS GIVEN BY THE POSITION EIGENSTATES

$$\{|\vec{x}\rangle\} \qquad (\hat{\vec{x}}|\vec{x}_0\rangle = \vec{x}_0|\vec{x}_0\rangle)$$

with $\langle\vec{x}|\vec{x}'\rangle = \delta(\vec{x}-\vec{x}') \quad \forall \vec{x}, \vec{x}' \in \mathbb{R}^3$; $\int d^3x\, |\vec{x}\rangle\langle\vec{x}| = 1$ (identity operator)

A GENERIC STATE $|f\rangle$ CAN BE EXPRESSED AS

$$|f\rangle = \int d^3x\, \langle\vec{x}|f\rangle\, |\vec{x}\rangle$$

WHERE $f(\vec{x}) = \langle\vec{x}|f\rangle$ IS THE WAVE FUNCTION ($\langle\vec{x}_0|\vec{x}\rangle = \delta(\vec{x}-\vec{x}_0)$ → you know where the particle is)

ANOTHER POSSIBLE BASIS IS THAT OF MOMENTA EIGENSTATES:

$$\{|\vec{p}\rangle\} \qquad (\hat{\vec{p}}|\vec{p}_0\rangle = \vec{p}_0|\vec{p}_0\rangle)$$

WITH

$$\langle\vec{p}|\vec{p}'\rangle = \delta(\vec{p}-\vec{p}') \quad \forall \vec{p}, \vec{p}' \in \mathbb{R}^3 ; \quad \int d^3p\, |\vec{p}\rangle\langle\vec{p}| = 1.$$

RECALL THAT:

$$\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi)^{3/2}} e^{+i\vec{x}\cdot\vec{p}}$$ (WAVE-FUNCTION WITH DEFINITE $\vec{p}$ = "PLANE WAVE")

$$\langle\vec{p}|\vec{p}'\rangle = \int d^3x\, \langle\vec{p}|\vec{x}\rangle\langle\vec{x}|\vec{p}'\rangle = \int \frac{d^3x}{(2\pi)^3} e^{+i(\vec{p}'-\vec{p})\vec{x}} = \delta(\vec{p}-\vec{p}') \;\#.$$

$|f\rangle$ CAN BE ALSO EXPRESSED AS:

$$|f\rangle = \int d^3p\, \langle\vec{p}|f\rangle |\vec{p}\rangle = \int d^3p\, a(\vec{p})|\vec{p}\rangle$$

OBVIOUSLY $|\langle\vec{x}|f\rangle|^2 d^3x$ IS THE PROB. TO FIND THE PARTICLE BETW. $\vec{x}$ and $\vec{x}+d^3\vec{x}$ FOR A MEAS. OF THE POSITION AT $t=0$

for $\psi(0,\vec{x}) = f(\vec{x})$

$|\langle\vec{p}|f\rangle|^2 d^3p$ IS THE PROB. THAT THE PARTICLE HAS A MOMENTUM BETWEEN $\vec{p}$ and $\vec{p}+d^3\vec{p}$ IF A MEAS. OF

Property: $A(t,\vec{p})$ is the Fourier transformation of $\psi(t,\vec{x})$

$$a(\vec{p}) = \langle \vec{p} | f \rangle = \int d^3x \, \langle \vec{p} | \vec{x} \rangle \langle \vec{x} | f \rangle =$$

$$= \int d^3x \, \frac{e^{-i\vec{p}\vec{x}}}{(2\pi)^{3/2}} f(\vec{x}) = \int \frac{d^3x}{(2\pi)^3} f(\vec{x}) \, e^{-i\vec{p}\vec{x}}$$