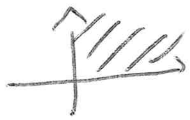
[illegible]

Ex. 1:

$$\begin{cases} f(x,y) = -K(x\ln x + y\ln y) \\ x + y = 1 \end{cases}$$

where $K > 0$.

$(x>0,\ y>0)$ is the domain



WAY 1:

Consider the parametric form of the constraint $x+y=1$:

$(x(t), y(t)) = (t, 1-t)$.

Then:

$$F(t) = f(x(t), y(t)) = -K\big(t\ln t + (1-t)\ln(1-t)\big).$$

$$\frac{dF}{dt} = -K\left(\ln t + 1 + (-1)\cdot\ln(1-t) + (\cancel{1-t})(-1)\cdot\frac{1}{\cancel{1-t}}\right)$$

$$= -K\big(\ln t + 1 - \ln(1-t) - 1\big) =$$

$$= -K\big(\ln t - \ln(1-t)\big) = 0$$

Ergo:

$$\ln t - \ln(1-t) = 0$$

$$\ln\frac{t}{1-t} = 0 \implies \frac{t}{1-t} = 1$$

$$t = 1 - t$$

$$2t = 1 \implies \boxed{t = 1/2}$$

It follows that the "extremum" is realised for the point

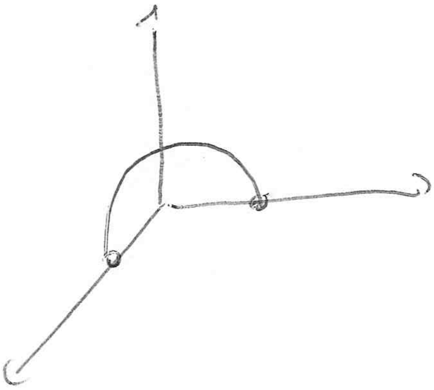
$(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

It is indeed a maximum. In fact:

$$\frac{d^2 F}{dt^2} = -K\left[\frac{1}{t} - (-1)\frac{1}{1-t}\right] = -K\left(\frac{1}{t} + \frac{1}{1-t}\right);$$

$$\left(\frac{d^2 F}{dt^2}\right)_{t=1/2} = -K\left(2+2\right) = -4K < 0$$

Then, $(x_0, y_0) = (1/2, 1/2)$ is a maximum on the constraint.



WAY 2: Lagrange multiplier.

$$g(x,y,\lambda) = -K(x\ln x + y\ln y) - \lambda(x+y-1)$$

$$\begin{cases} \partial_x g = -K(\ln x + 1) - \lambda = 0 \\ \partial_y g = -K(\ln y + 1) - \lambda = 0 \\ \partial_\lambda g = x + y - 1 = 0 \end{cases}$$

out of the first 2:

$$\ln x + 1 = \ln y + 1 \Rightarrow \ln x = \ln y \rightarrow x = y$$

out of the the last eq:

$$2x - 1 = 0 \rightarrow x = 1/2$$

Ergo $\left(\frac{1}{2}, \frac{1}{2}\right)$ is the extremum.

Ex. 2

2.1)

$$f(x,y) = (x^2-y^2)^2 - \alpha x^2 y^2$$

$$\partial_x f(x,y) = 2x \cdot 2(x^2-y^2) - \alpha \cdot 2x \cdot y^2 = 4x(x^2-y^2) - 2\alpha x \text{[illegible]}$$

$$\partial_y f(x,y) = -2y \cdot 2 \cdot (x^2-y^2) - \alpha x^2 \cdot 2y = -4y(x^2-y^2) - 2\alpha x^2 \text{[illegible]}$$

$$\partial_x^2 f(x,y) = 4(x^2-y^2) + 4x \cdot 2x - 2\alpha y^2$$

$$\partial_y^2 f(x,y) = -4(x^2-y^2) - 4y(-2y) - 2\alpha x^2$$

$$(\partial_x^2 + \partial_y^2) f = 0$$

$$\cancel{4(x^2-y^2)} + 8x^2 - 2\alpha y^2 - \cancel{4(x^2-y^2)} + 8y^2 - 2\alpha x^2 = 0$$

$$8(x^2+y^2) - 2\alpha(x^2+y^2) = 0$$

Ergo:

$$\boxed{\alpha = 4}$$

It is important to notice that this solution can be not WRITEN as $U(x)\,W(y)$! It does not belong to the solutions that one can obtain via the separation Ansatz.

2.2)

$$\left(\lambda\partial_t - \partial_x^2\right)\phi(t,x) = 0$$

$$\phi(t,x) = U(x)\,W(t)$$

Put in:

$$\left(\lambda\partial_t - \partial_x^2\right)U(x)\,W(t) = U(x)\left[\lambda\partial_t W(t)\right] - \left[\partial_x^2 U(x)\right]W(t) = 0$$

$$\frac{\lambda\partial_t W(t)}{W(t)} = \frac{\partial_x^2 U(x)}{U(x)} = K = \text{const.}$$

$$\frac{\lambda\,\partial_t W(t)}{W(t)} = K$$

We have the soln

$$\lambda\frac{dW}{dt} = KW \Rightarrow \frac{dW}{W} = \frac{K}{\lambda}\,dt \Rightarrow \frac{W}{W_0} = e^{\frac{K}{\lambda}t}$$

$$W(t) = e^{\frac{K}{\lambda}t}\cdot W_0$$

check:

$$\frac{dW}{dt} = \frac{K}{\lambda}e^{\frac{K}{\lambda}t}W_0 \Rightarrow \lambda\frac{dW}{dt}\cdot\frac{1}{W} = K, \text{ OK}$$

✓

$$\frac{d^2U(x)}{dx^2} = KU^2(x)$$

$$\frac{d^2U(x)}{dx^2} - KU^2(x) = 0$$

For $K<0$ the solution is a oscillating function:

$$U(x) = A\cos(\sqrt{-K}\,x) + B\sin(\sqrt{-K}\,x)$$

Ergo, the final solution for $K<0$ is:

$$\phi(t,x) = \left(A\cos(\sqrt{-K}\,x) + B\sin(\sqrt{-K}\,x)\right)\cdot\left(W_0 e^{\frac{K}{\lambda}t}\right)$$

whereas $K<0$

and A, B, W_0, K are integration constants.

Initial condition:

$$\phi(0,x) = \cos(2x)$$

From the general solution:

$$\phi(0,x) = A W_0 \cos(\sqrt{-K}\, x) + B W_0 \sin(\sqrt{-K}\, x)$$

It then follows that:

$$A W_0 = 1 \rightarrow A = \frac{1}{W_0}$$

$$B W_0 = 0 \rightarrow B = 0$$

$$\sqrt{-K} = 2 \rightarrow K = -4$$

Then, the general solution for this initial condition is:

$$\boxed{\phi(t,x) = \cos(2x)\, e^{-\frac{4}{\lambda}t}}$$

2.3)

$$\left(\frac{\partial_t^2}{c^2} - \partial_x^2 \right) \phi(t,x) = 0$$

$$\phi(t,x) = W(t)\, U(x)$$

$$\frac{\partial_t^2 W(t)}{c^2} \cdot U(x) - W(t)\, \partial_x^2 U(x) = 0$$

$$\frac{1}{c^2} \frac{\partial_t^2 W(t)}{W(t)} = \frac{\partial_x^2 U(x)}{U(x)} = K = \text{const.}$$

Now, for $K > 0$

$$\frac{\partial_x^2 U(x)}{U} = K$$

$$\frac{d^2 U(x)}{dx^2} - K U(x) \rightarrow U(x) = \alpha e^{\sqrt{K} x} + \beta e^{-\sqrt{K} x}$$

$$\frac{1}{c^2} \frac{\partial_t^2 W(t)}{W(t)} = K \rightarrow W(t) = \gamma e^{\sqrt{K} c t} + \delta e^{-\sqrt{K} c t}$$

Put together:

$$\boxed{\phi(t,x) = W(t)\, U(x) = \alpha\gamma e^{\sqrt{K}(x+ct)} + \alpha\delta e^{\sqrt{K}(x-ct)} + \beta\gamma e^{-\sqrt{K}(x-ct)} + \beta\delta e^{-\sqrt{K}(x+ct)}}$$

We have then shown that

$$\phi(t,x) = f(x-ct) + g(x+ct)$$

with

$$f(x-ct) = \alpha\delta e^{\sqrt{K}(x-ct)} + \beta\gamma e^{-\sqrt{K}(x-ct)}$$

$$g(x+ct) = \alpha\gamma e^{\sqrt{K}(x+ct)} + \beta\delta e^{-\sqrt{K}(x+ct)}$$

For $K < 0$ we get

$$\begin{cases} u(x) = \alpha\cos(\sqrt{-K}\,x) + \beta\sin(\sqrt{-K}\,x) \\ w(t) = \gamma\cos(\sqrt{-K}\,ct) + \delta\sin(\sqrt{-K}\,ct). \end{cases}$$

Repeat the steps and use the bisection formulas ....

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\sin(a+b) = \cos a\sin b + \cos b\sin a$$

...

...

3)

$M = \{a, b\}$

We need 1_G.

We assume that $a = 1_G$. Then:

$$\begin{cases} a \cdot a = a \\ a \cdot b = b \cdot a = b \end{cases}$$

a is the "unity" and is the inverse of itself:

$a^{-1} = a$ belong to M.

Now, b needs an inverse. Considering that no other elements are present, it must be that

$$b \cdot b = a = 1_G.$$

$$\boxed{b = b^{-1}}$$

Ergo:

$$\begin{cases} a \cdot a = a \\ a \cdot b = b \cdot a = b \\ b \cdot b = a \end{cases}$$

with this operation $(M, \cdot)$ is a group.

A simple representation of this group is:

$M = \{1, -1\}$ $(a = 1, b = -1)$

with $\cdot$ = usual multiplication.

In fact:

$a \cdot a = 1 \cdot 1 = 1$

$a \cdot b = b \cdot a = 1(-1) = (-1) \cdot (1) = -1 = b$

$b \cdot b = (-1)(-1) = 1$

Note, this group turns out to be an abelian group!!!