Ex. 1

1.1) $xy' - 1 = 0$

$x\,y'(x) = 1 \qquad y'(x) = \frac{1}{x}$

General solution:

$\boxed{y(x) = \ln x + C}$

With the initial condition $y(1) = 1$ we get

$y(1) = \ln 1 + 1 = 1$

$\boxed{y(x) = \ln x + 1 = \ln(ex)}$

Check:

$y'(x) = \frac{1}{x}$ ✓

1.2) $y' - x^2 y = 0$

$\int \frac{dy}{y} = x^2 dx$

Integrate:

$\ln y = \frac{x^3}{3} + C \quad \rightarrow \quad \boxed{y(x) = e^{\frac{x^3}{3} + C} = K e^{x^3/3}}.$

General solution.

$y(1) = 1$

$$y(1) = Ke^{1/3} = 1 \rightarrow K = \frac{1}{e^{1/3}} = e^{-1/3}.$$

$$\boxed{y(x) = e^{\frac{x^3}{3} - \frac{1}{3}}}$$

$y(1)=1$ is fulfilled.

1.3) $y' - \frac{4}{y} y = -x^3$

$$p(x) = -\frac{4}{x} \Rightarrow e^{\int -\frac{4}{x} dx} = e^{\ln x^{-4}} = \frac{1}{x^4} = e^{\int p(x) dx}$$

$$e^{-\int p(x) dx} = e^{\ln x^4} = x^4$$

$$\int q(x) e^{\int p\, dx} dx = \int (-x^3) \frac{1}{x^4} dx = \int -\frac{1}{x} dx = -\ln x.$$

Ergo, the most general solution is:

$$\boxed{y(x) = x^4 \left(-\ln x + c \right).}$$

A simple check shows that the diff. eq. is solved for each c.

Taking into account the initial condition:

$$y(e) = 0$$

$$y(e) = e^4 \left[-\ln e + c \right] = 0 \quad \rightarrow \quad c = \ln e = 1.$$

We then obtain the particular unique solution fulfilling $y(e) = 0$ for

$$\boxed{y(x) = x^4 \left(-\ln x + 1 \right)}$$

x.2

2.1) $y'' + a^2 y = 0$

The most general solution is:

$$\boxed{y = A\sin(ax) + B\cos(bx)}$$

$(y_1 = \sin(ax)$ and $y_2 = \cos(bx)$ are two independent solutions forming a basis for the most general solution$)$.

Check:

$$y' = Aa\cos(ax) - Bba\sin(ax)$$

$$y'' = -Aa^2\sin(ax) - Ba^2\cos(ax) = -a^2 y.$$

initial conditions:

$$\begin{cases} y(0) = 0 \\ y'(0) = 4 \end{cases}$$

$y(0) = 0 \rightarrow B = 0$

$y'(0) = aA = 4 \rightarrow A = 4/a$

Unique solution: $\boxed{y = \frac{4}{a}\sin(ax)}$

22) $y'' - a^2 y = b$.

The solutions of the homog. equations are:

$$y_1 = e^{ax}$$

$$y_2 = e^{-ax}$$

The hom. sol. is:

$$y_{hom} = A e^{ax} + B e^{-ax} \quad \rightarrow \text{check: } y''_{hom} - a^2 y_{hom} = 0$$

A special solution is given by:

$$y_{spec}(x) = -\frac{b}{a^2}. \quad \left[y'_{spec}(x) = 0 \,,\, y''_{spec}(x) = 0 \rightarrow -a^2\left(-\frac{b}{a^2}\right) = b \checkmark \right]$$

Then, the most general solution is:

$$y(x) = \underbrace{A e^{ax} + B e^{-ax}}_{y_{hom}} + \underbrace{-\frac{b}{a^2}}_{y_{spec}}$$

Initial condition:

$$\begin{cases} y(0)=0 & \longrightarrow A+B-\frac{b}{a^2}=0 \\ y'(0)=0 & \longrightarrow A-B=0 \rightarrow A=B \end{cases}$$

Put the 2° into the 1°:

$$A=B=\frac{1}{2}\frac{b}{a^2}$$

$$\boxed{y(x)=\frac{1}{2}\frac{b}{a^2}\left(e^{ax}+e^{-ax}\right)-\frac{b}{a^2}}$$

2.3) $y''-\lambda y'=0$

Put: $y'=u \quad y''=u'$

$$u'-\lambda u=0 \Longrightarrow \ln u=\lambda x+\text{const} \rightarrow u=ke^{\lambda x}$$

Ergo:

$$y'=u=ke^{\lambda x}$$

$$\boxed{y=\frac{k}{\lambda}e^{\lambda x}+c}$$ general solution

Put the initial conditions:

$$\begin{cases} y(0)=0 \rightarrow \frac{k}{\lambda}+c=0 \rightarrow & c=-\frac{1}{\lambda} \\ y'(0)=1 \rightarrow k e^{\lambda \cdot 0} = k = 1 \end{cases}$$

Ergo:

$$\boxed{y(x) = \frac{1}{\lambda} e^{+\lambda x} - \frac{1}{\lambda}}$$

2.5) $y''' = 0$.

$y_1 = 1$

$y_2 = x$

$y_3 = x^2$

Most general solution:

$$\boxed{y(x) = A + Bx + Cx^2}$$ (easy to check).

Initial conditions

$y(0) = 0$

$y(1) = 0$

$y(2) = 1$

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You get:

$y(0) = A = 0 \rightarrow A = 0$

$y(1) = B + C = 0 \rightarrow B = -C$

$y(2) = 2B + 4C = 2 \longrightarrow 2C = 1 \quad C = \frac{1}{2}$

Ergo:

$$\boxed{C = \frac{1}{2}, \; B = -\frac{1}{2}, \; A = 0}$$

Put together:

$$\boxed{y(x) = \frac{1}{2} x(x-1) = -\frac{1}{2} x + \frac{1}{2} x^2}$$