2.1) $A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$

$$\det(A - \lambda 1_2) = (2-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$\boxed{\lambda_1 = 1}$

$\boxed{\lambda_2 = 4}$

2.2)

$\vec{v}_1$ /

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 = \vec{v}_1$$

$$\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x + \sqrt{2}\, y = 0 \to \boxed{y = -\frac{1}{\sqrt{2}} x}$

The 2nd eq. gives us the same condition...

Then, we require that $y^2 + x^2 = 1$.

$$\frac{1}{2}x^2 + x^2 = 1 \;;\; \frac{3}{2}x^2 = 1 \to x = \sqrt{\frac{2}{3}}$$

Ergo:

$$\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$ is the eigenv. with $\lambda_1 = 1$

check:

$$\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{\sqrt{3}} \\ \sqrt{2}\sqrt{\frac{2}{3}} - \frac{3}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} \\ -1/\sqrt{3} \end{pmatrix}$$ i.e. $\lambda_1 = 1$!

$\equiv$

$\lambda_2 = 4$

$$\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-2x + \sqrt{2}\, y = 0 \implies y = \frac{2}{\sqrt{2}} x = \sqrt{2}\, x$

$x^2 + y^2 = 1 \qquad x^2 + 2x^2 = 1 \to x = \frac{1}{\sqrt{3}}$

Ergo:

$$\vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$ is the Eigenvector with eigenvalue $\lambda_2 = -1$.

2.3) $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^{\,t}\,\vec{v}_2 = \left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)\begin{pmatrix}\sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}}\end{pmatrix} = \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} = 0$!!

2.4

$$B = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$B^t B = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ +\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}+\frac{1}{3} & \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The same for BB^t.

2.5)

$B^t A B$

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2\sqrt{2}-\sqrt{2} & 2+2 \\ 2-3 & \sqrt{2}+3\sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 4 \\ -1 & 4\sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} 2+1 & 4\sqrt{2}-4\sqrt{2}=0 \\ \sqrt{2}-\sqrt{2}=0 & 4+4\cdot 2 \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Yes... it is exactly the expected result being

$$B^t A B = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

2.6)

$\det A = 3\cdot 2 - 2 = 6-2 = 4 = \lambda_1 \cdot \lambda_2 = 1\cdot 4 = 4$

$\mathrm{Tr} A = 3+2 = 5 = \lambda_1 + \lambda_2 = 1+4 = 5$!!!

Both eqs are fullfilled!!!!!

This is also expected being in general

$$\begin{cases} \det A = \lambda_1 \cdots \lambda_N \\ \mathrm{Tr} A = \lambda_1 + \cdots + \lambda_N \end{cases}$$

2.7)

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -1 \\ \frac{1}{4} & \frac{\sqrt{2}}{4} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2+\frac{1}{4} & -\sqrt{2}+\frac{\sqrt{2}}{4} \\ -\sqrt{2}+\frac{\sqrt{2}}{4} & 1+\frac{2}{4} \end{pmatrix} =$$

c.t

$$= \frac{1}{3}\begin{pmatrix} \frac{9}{4} & \sqrt{2}\left(-1+\frac{1}{4}\right) \\ \sqrt{2}\left(-1+\frac{1}{4}\right) & \frac{3}{2} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{9}{4} & \sqrt{2}\,\frac{3}{4} \\ \sqrt{2}\,\frac{3}{4} & \frac{3}{2} \end{pmatrix} =$$

$$= \boxed{\begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix} = M}$$

$$MA = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{6}{4} - \frac{2}{4} & \frac{3\sqrt{2}}{4} - \frac{3\sqrt{2}}{4} = 0 \\ -\frac{2\sqrt{2}}{4} + \frac{\sqrt{2}}{2} = 0 & -\frac{\sqrt{2}}{4}\sqrt{2} + \frac{3}{2} = 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Yes, it is expected... In fact:

$A^{-1} = M = B D^{-1} B^t$, see the last ex. of Blatt 9!!

3

$$e^{\varphi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

$$1 + \varphi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + \frac{\varphi^2}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^2 + \cdots =$$

$$= \sum_{n=0}^{\infty} \frac{\varphi^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{\varphi^{2n}}{(2n)!} A^{2n} + \sum_{n=0}^{\infty} \frac{\varphi^{2n+1}}{(2n+1)!} A^{2n+1} =$$

$$= \sum_{n=0}^{\infty} \frac{\varphi^{2n}}{(2n)!} (-1)^n \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sum_{n=0}^{\infty} \frac{\varphi^{2n+1}}{(2n+1)!} (-1)^n \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\varphi^{2n}}{(2n)!} (-1)^n & \sum_{n=0}^{\infty} \frac{\varphi^{2n+1}}{(2n+1)!} (-1)^n \\ -\sum_{n=0}^{\infty} \frac{\varphi^{2n+1}}{(2n+1)!} (-1)^n & \sum_{n=0}^{\infty} \frac{\varphi^{2n}}{(2n)!} (-1)^n \end{pmatrix} =$$

$$= \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

e) 4

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\det A = 3 \cdot (4-5) + 1 \cdot (2-4) = -3 - 2 = -5 \neq 0!$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}; \quad \det A_1 = 1 \cdot (1-8) = -7$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{-7}{-5} = \frac{7}{5}$$

$$A_2 = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ 1 & 2 & 1 \end{pmatrix}; \quad \det A_2 = 3(1-10) + 1(4-1) =$$

$$= -27 + 3 = -24$$

$$y = \frac{\det A_2}{\det A} = \frac{24}{5}$$

$$A_3 = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\det A_3 = 3(8-1) = 21$$

$$z = \frac{\det A_3}{\det A} = -\frac{21}{5}$$

Verify:

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 7/5 \\ 24/5 \\ -21/5 \end{pmatrix} = \begin{pmatrix} \frac{21}{5} - \frac{21}{5} = 0 \\ \frac{14}{5} + \frac{24\cdot 4}{5} - \frac{21\cdot 5}{5} = \frac{5}{5} = 1 \\ \frac{7}{5} + \frac{24}{5} - \frac{21}{5} = \frac{10}{5} = 2 \end{pmatrix}$$