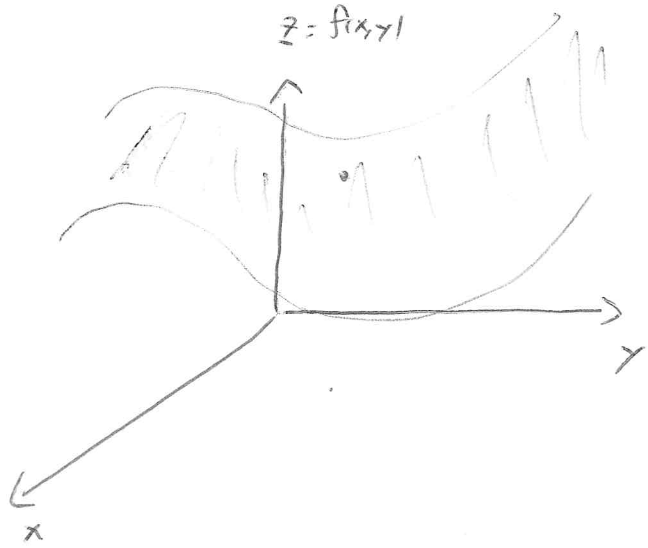
$$f(x,y): \mathbb{R}^2 \mapsto \mathbb{R}$$

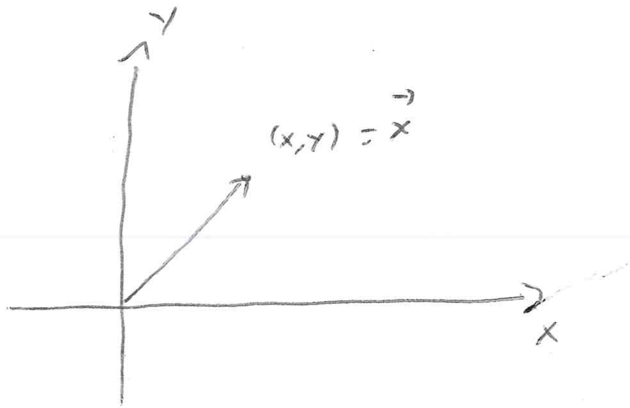
Examples:

$f(x,y) = x + y$

$f(x,y) = x^3 y^4 + \sin(xy)$

...



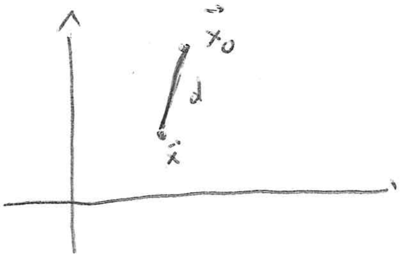


One writes also

$f(\vec{x})$. This is a "scalar field".

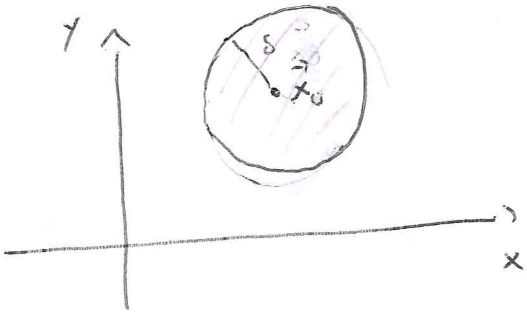
[The generalisation to $f(x_1, \dots, x_N): \mathbb{R}^N \mapsto \mathbb{R}$ is straightforward but cannot be any longer visualised... $f: \mathbb{R}^2 \mapsto \mathbb{R}$ is the "limit"...]

In R^2 we can also define the 'distance' of two points $\vec{x} = (x, y)$ and $\vec{x}_0 = (x_0, y_0)$:

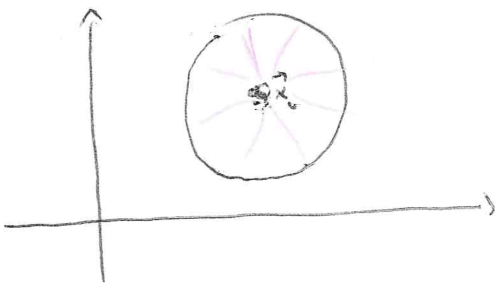


$$d(\vec{x}, \vec{x}_0) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$I(\vec{x}_0; \delta) \overset{def}{=} \left\{ \vec{x} \in R^2 \text{ such that } d(\vec{x}, \vec{x}_0) < \delta \right\}$$



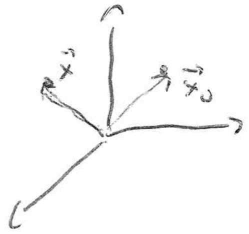
$$\dot{I}(\vec{x}_0; \delta) \overset{def}{=} \left\{ \vec{x} \in R^2 \text{ such that } d(\vec{x}, \vec{x}_0) < \delta \text{ and } \vec{x} \neq \vec{x}_0 \right\}$$



$\vec{x}_0$ is not part of $\dot{I}(\vec{x}_0; \delta)$

Again, the generalisation to the case R^N with $\vec{x} = (\vec{x}_1, \dots, \vec{x}_N)$ is simple.

In fact, for R^N we have:



$\vec{x} = (x_1, x_2, \dots, x_N)$

$\vec{x}_0 = (x_{0,1}, x_{0,2}, \dots x_{0,N})$

one has:

$$d(\vec{x}, \vec{x}_0) = \sqrt{(x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2 + \dots (x_N - x_{0,N})^2}$$

Note that for the case $N = 1$ (i.e., the old one dimensional case) we reobtain

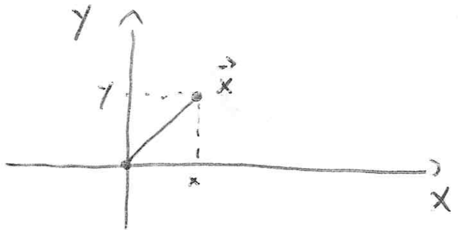
$$d(x, x_0) = \sqrt{(x - x_0)^2} = |x - x_0|$$

$I(x_0, \delta)$

$\dot{I}(x_0, \delta)$

A 'trivial' consideration:

$d(\vec{x},\vec{0})$ is the length of the vector $\vec{x}$.



$$d(\vec{x},0) = \ell(\vec{x}) = \|\vec{x}\| = \sqrt{x^2+y^2}$$

In N dimensions $(\vec{x} = (x_1, \dots, x_N))$

$$d(\vec{x},0) = \ell(\vec{x}) = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

One can repeat the steps done for the case $f: R \mapsto R$ and study:

limits, derivatives, integrals and differential equations.

of course, things become more difficult when more dimensions is present.

$$\lim_{\vec{x} \mapsto \vec{x}_0} f(\vec{x}) = L \qquad f(\vec{x}) = f(x,y) = R^2 \mapsto R.$$

$\forall \varepsilon > 0 \ \exists \delta > 0$ such that for $\vec{x} \in \dot{I}(\vec{x}_0, \delta) \rightarrow | f(\vec{x}) - L | < \varepsilon$

(The definition is indeed valid also for the general N case and for $N=1$ it reduces to the definition of limit we have discussed in the first lecture)

In the case $N=2$ one writes also

$$\lim_{(x,y) \mapsto (x_0,y_0)} f(x,y) = L$$

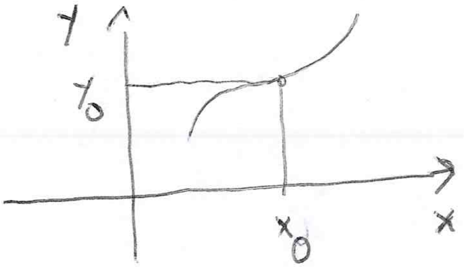
$$\begin{cases} \lim\limits_{(x,y)\mapsto(0,0)} (x^2+y^2) = 0 \\ \lim\limits_{(x,y)\mapsto(0,0)} (x^2\cdot y^2) = 0 \end{cases} \qquad \begin{cases} \lim\limits_{(x,y)\mapsto(1,1)} (x^2+y^2) = 2 \\ \lim\limits_{(x,y)\mapsto(1,1)} x^2 y^2 = 1\cdot 1 = 1 \end{cases}$$

Note that, when the limit exists, it is allowed to perform the limit separately:

$$\lim_{(x,y)\mapsto(x_0,y_0)} f(x,y) = \lim_{x\mapsto x_0}\left(\lim_{y\mapsto y_0} f(x,y)\right) = \lim_{y\mapsto y_0}\left(\lim_{x\mapsto x_0} f(x,y)\right).$$

More in general, consider a function $y(x)$

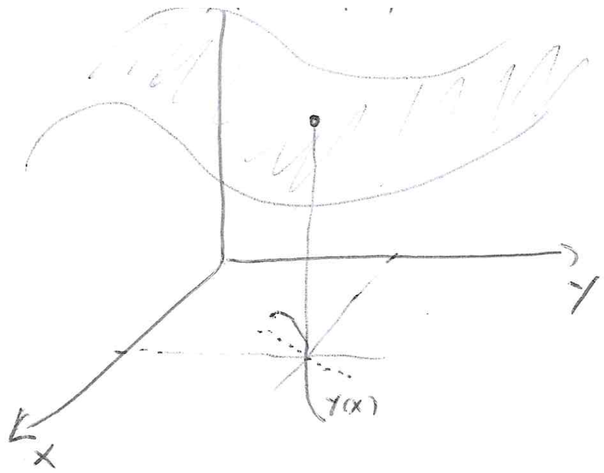
$y = y(x)$ / $\lim\limits_{x\mapsto x_0} y(x) = y_0$ (i.e., $y(x)$ is a cont. function)



Then, let $F(x) = f(x, y(x))$:

$$\lim_{x\mapsto x_0} F(x) = \lim_{\vec{x}\mapsto\vec{x}_0} f(x,y) = L \quad \text{for each function } y(x).$$

Exemple



The value of the limit is indipendent on the choosen direction....

It does not depend on the way we approach the point (x_0, y_0).

$$\lim_{(x,y)\mapsto(1,1)} x^2 y^2$$

Let us consider

$$y(x) = x^{10} \quad ; \quad \lim_{x\mapsto 1} y(x) = 1$$

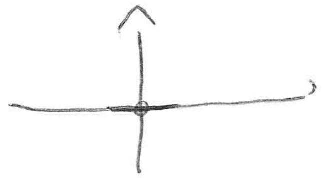
$$F(x) = x^2 \cdot x^{10} = x^{12}$$

$$\lim_{x\mapsto 1} F(x) = \lim_{(x,y)\mapsto(1,1)} x^2 y^2 = 1.$$

However, not always the limit exists (even if it may exist and be well defined in some directions).

To this end let us study the function

$$f(x,y) = \frac{|x\,y|}{x^2+y^2}$$



Let us consider first

$y(x) = 0 \qquad \lim_{x \mapsto 0} y(x) = 0$

$f(x,0) = 0 \quad \forall x \neq 0$

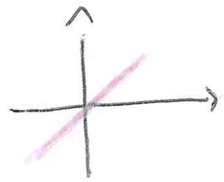
Ergo, we get for $F(x, y(x)) = \frac{|x\,y(x)|}{x^2 + y^2(x)} = 0 \quad$ for $x \neq 0$

it follows that along this direction

$\lim_{x \mapsto 0} F(x) = 0$

However, if we consider $y(x) = x$ (for which it also holds

$\lim_{x \mapsto 0} y(x) = \lim_{x \mapsto 0} x = 0$



$\Rightarrow G(x) = f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \quad \forall x \neq 0$

$\lim_{x \mapsto 0} G(x) = 1/2$

it is then clear that the "limit" does <u>not</u> exist.

(Along the direction $y=0$ the limit exists and is zero

" " " $y=x$ " " " " " $\frac{1}{2}$

$\Downarrow$

No overall limit is defined)

A function $f(\vec{x}) : \mathbb{R}^2 \mapsto \mathbb{R}$ is continous in $\vec{x}_0$ if

$$\lim_{\vec{x} \mapsto \vec{x}_0} f(\vec{x}) = f(\vec{x}_0) .$$

Examples:

- $f(x, y) = \sqrt{|xy|}$ is cont. in $\mathbb{R}^2$ (everywhere).
- $f(x,y) = \log(x^4 + y^4)$ is cont. for $(x,y) \neq (0,0)$.
- $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

Derivatives

In the case of two (or more) dimensions one defines "partial derivatives":

$$f(x,y) : \mathbb{R}^2 \mapsto \mathbb{R}$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \mapsto 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\left(\frac{\partial f(x,y)}{\partial x}\right)\Bigg|_{\vec{x}=\vec{x}_0} = \lim_{h \mapsto 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

Similarly:

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \mapsto 0} \frac{f(x,y+h) - f(x,y)}{h}$$

One then introduces the gradient

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

[The gradient has a "precise" geometrical meaning, but we describe it afterwards...]

Example :

$$f(x,y) = x^7 \sin(y)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 7x^6 \sin(y) \\ \dfrac{\partial f}{\partial y} = x^7 \cos(y) \end{cases}$$

one can go further and study the second derivatives :

$$\frac{\partial^2 f}{\partial x^2} = 7 \cdot 6\, x^5 \sin(y)$$

$$\frac{\partial^2 f}{\partial y^2} = -x^7 \sin y$$

But there are also mixed derivatives :

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(7x^6 \sin y\right) = 7x^6 \cos y$$

$$\parallel$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^7 \cos y\right) = 7x^6 \cos y$$

This is not a coincidence...

The equality $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$ holds in general. It can be easily proven by using properties of the limits.

A Taylor expansion can be performed also in more dimensions:

$$f(x,y) = a_{00} + a_{10}\, x + a_{01}\, y + a_{11}\, x y + a_{20}\, x^2 + a_{02}\, y^2$$

$$+ a_{21}\, x^2 y + \cdots =$$

$$= \sum_{n,m} a_{nm}\, x^n y^m .$$

$a_{00} = f(0,0).$

$$\frac{\partial f}{\partial x} = a_{10} + a_{11}\, y + \cdots$$

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=0\\ y=0}} = a_{10}$$

$$\left(\frac{\partial f}{\partial y}\right)_{\substack{x=0\\ y=0}} = a_{01}$$

$$\left(\frac{\partial f}{\partial x \partial y}\right)_{\substack{x=0\\ y=0}} = a_{11} = \left(\frac{\partial f}{\partial y \partial x}\right)_{\substack{x=0\\ y=0}}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=0\\y=0}} = 2a_{20}$$

$$a_{20} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^2 f}{\partial x^2}\left(a_{21} x^2 + \dots\right) = 2a_{21} \dots$$

In general:

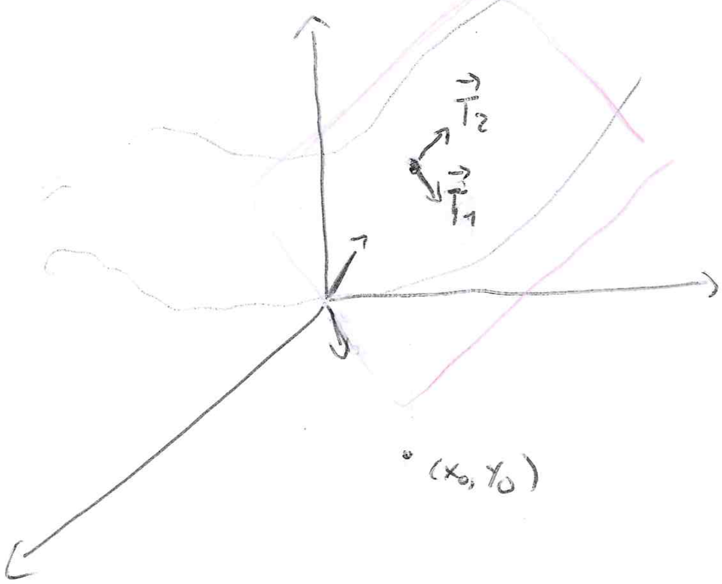
$$\boxed{a_{nm} = \frac{1}{n!}\frac{1}{m!}\left[\frac{\partial^{n+m} f}{(\partial x)^n (\partial y)^m}\right]_{\substack{x=0\\y=0}}}$$

In general, choosing the point $\vec{x}_0 = (x_0, y_0)$ we repeat all the steps and get

$$f(x,y) = a_{00} + a_{10}(x-x_0) + a_{01}(y-y_0) + a_{20}(x-x_0)^2 + a_{02}(y-y_0)^2$$
$$+ a_{11}(x-x_0)(y-y_0) + \dots$$

$$a_{nm} = \frac{1}{n!\,m!}\left[\frac{\partial^{n+m} f}{(\partial x)^n (\partial y)^m}\right]_{\substack{x=x_0\\y=y_0}}$$

Again, the gener. to N dimension is straightforward



$\vec{T}_1$ and $\vec{T}_2$ are the tangent-vectors. How to determine them?

A Taylor expansion of the function up to the first order gives

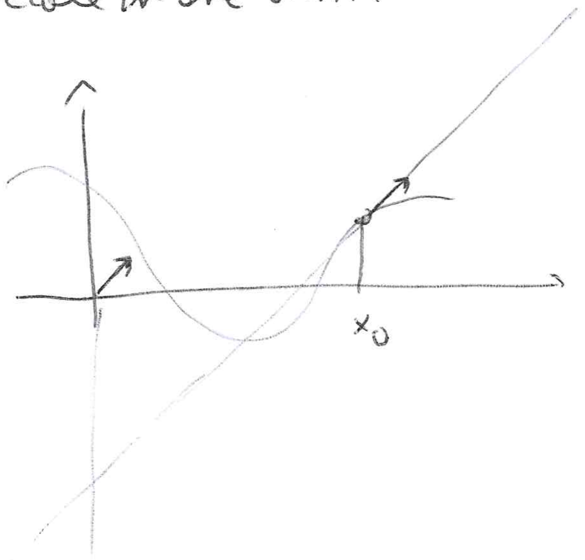
$$\underbrace{f(x,y) \simeq f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_{x_0,y_0}(x-x_0) + \left(\frac{\partial f}{\partial y}\right)_{x_0,y_0}(y-y_0)}_{\text{this is the tg plane}}$$

The vectors $\vec{T}_1$ and $\vec{T}_2$ which are

$$\vec{T}_1 = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2_{\vec{x}_0}}}\left(1, 0, \left(\frac{\partial f}{\partial x}\right)_{\vec{x}_0}\right) \qquad \ell(\vec{T}_1) = 1.$$

$$\vec{T}_2 = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial y}\right)^2_{\vec{x}_0}}}\left(0, 1, \left(\frac{\partial f}{\partial y}\right)_{\vec{y}_0}\right)$$

Recall in one dim:



$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

The tg-vector is:

$$\vec{T} = \frac{1}{\sqrt{1 + f'(x_0)^2}}\,(1, f'(x_0))$$

In N dimensions I have $f(x_1, \dots, x_N)$. One has N tg vectors

$$\vec{x}_0 = (x_{0,1}, \dots x_{0,N})$$

$$\begin{cases} T_1 = \dfrac{\left(1, 0, \dots, 0, \left(\frac{\partial f}{\partial x_1}\right)_{\vec{x}_0}\right)}{\sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2_{\vec{x}_0}}} \\ \dots \\ \vec{T}_{N-1} = \left(0, \dots, 1, \left(\frac{\partial f}{\partial x_N}\right)_{\vec{x}_0}\right) \Big/ \sqrt{1 + \left(\frac{\partial f}{\partial x_N}\right)^2_{\vec{x}_0}} \end{cases}$$

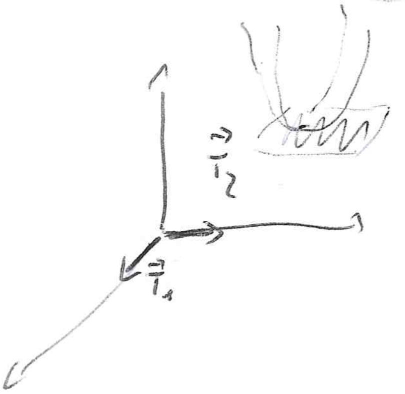
If the 1° derivatives are zero one has a maximum or a minimum or a "saddle point".

$$\left(\frac{\partial f}{\partial x}\right)_{\vec{x}_0} = 0 \,, \quad \left(\frac{\partial f}{\partial y}\right)_{\vec{x}_0} = 0$$

i.e.: $\left(\vec{\nabla} f\right)_{\vec{x}_0} = \left(\left.\frac{\partial f}{\partial x}\right|_{\vec{x}_0}, \left.\frac{\partial f}{\partial y}\right|_{\vec{x}_0}\right) = \vec{0} = (0,0)$

$\vec{T}_1 = (1,0,0)$

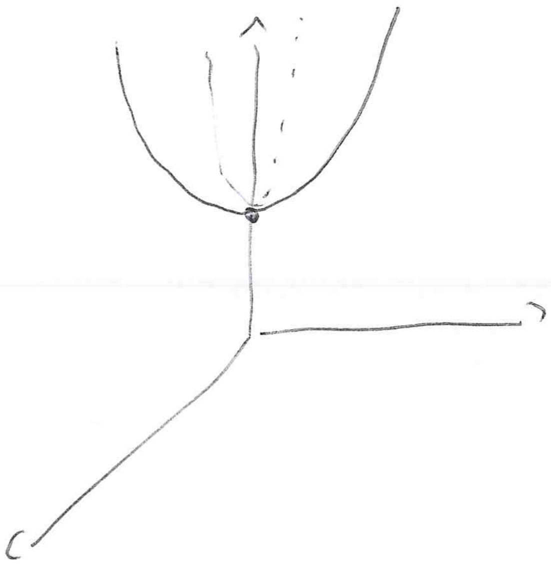
$\vec{T}_2 = (0,1,0)$



The tg plane is parallel to the xy-plane.

<u>Examples</u>:

$$f(x,y) = x^2 + y^2 + 1$$



$\vec{x}_0 = (0,0)$

$$\frac{\partial f}{\partial x} = 2x = 0 \rightarrow x = 0$$

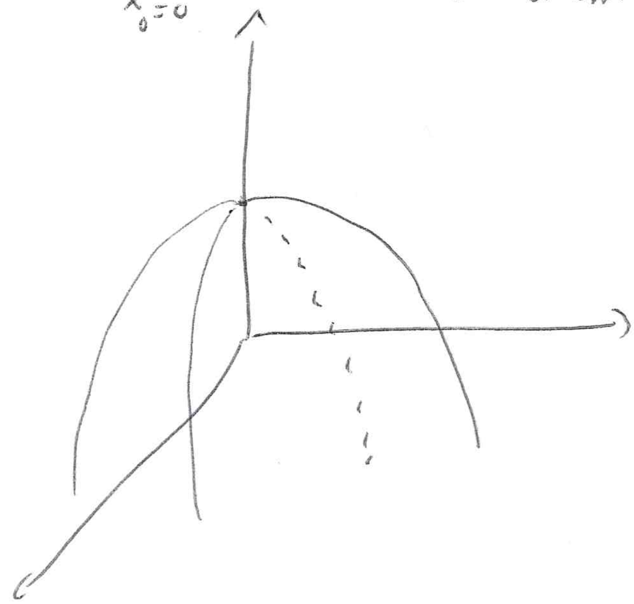
$$\frac{\partial f}{\partial y} = 2y \qquad \rightarrow \left(\vec{\nabla} f\right)_{\vec{x}_0 = \vec{0}} = (0,0) = \vec{0} \;!$$

The point $\vec{x}_0 = (0,0)$ is a minimum.

Now, if we have

$$f(x,y) = -x^2 - y^2 + 1$$

$(\vec{\nabla} f)_{\vec{x}_0 = \vec{0}} = (0,0)$ ... it is still an extremum ....



The point

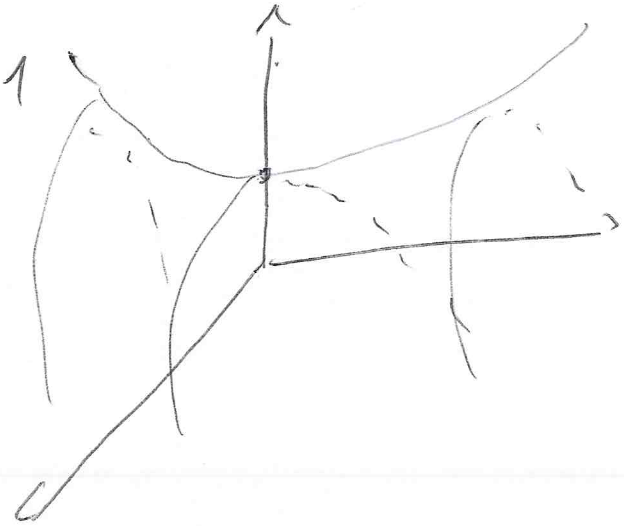
$$P = (0,0,1)$$

is a maximum!

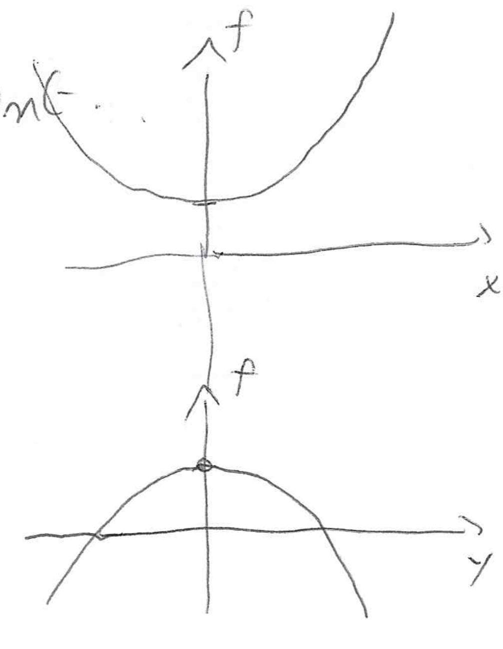
But there is a third possibility:

$$f(x,y) = x^2 - y^2 + 1$$

$$(\vec{\nabla} f)_{\vec{x} = \vec{0}} = (0,0)$$



This is a saddle point...



plane $y = 0$

plane $x = 0$

But what if we have:

$$f(x,y) = x^2 + y^2 - xy \; ?$$

$$\frac{\partial f}{\partial x} = 2x - y \quad = 0 \quad \text{for } x = y = 0$$

$$\frac{\partial f}{\partial y} = 2y - x \quad = 0 \quad \text{'' ......}$$

it is maximum, minimum or saddle point?

is there a way how to study it?

Indeed yes, but it goes a bit "beyond" the present status our knowledge.

One considers the so-called Hesse matrix

$$H = \begin{pmatrix} \left.\frac{\partial^2 f}{\partial x^2}\right|_{\vec{x}_0} & \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\vec{x}_0} \\ \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{\vec{x}_0} & \left.\frac{\partial^2 f}{\partial y^2}\right|_{\vec{x}_0} \end{pmatrix}$$

and studies the "eigenvalues of it".

compared to [illegible]

$$H = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

$$\det(H - \lambda 1) = \det\begin{pmatrix} a-\lambda & c \\ c & b-\lambda \end{pmatrix} = (a-\lambda)(b-\lambda) - c^2 = 0.$$

$\Rightarrow \lambda_1, \lambda_2$ are the two eigenvalues.

$\lambda_1 > 0, \lambda_2 > 0 \longrightarrow$ Minimum.

$\lambda_1 < 0, \lambda_1 < 0 \mapsto$ Maximum

$\lambda_1 \cdot \lambda_2 < 0 \rightarrow$ Saddle.

if $\lambda_1 = 0$ or $\lambda_2 = 0 \rightarrow$ Nothing can be said

Let us see the previous examples:

$$f = x^2 + y^2 + 1$$

$$\frac{\partial f}{\partial x} = 2x \rightarrow \frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial f}{\partial y} = 2y \rightarrow \frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(2-\lambda)(2-\lambda) = 0$$

$$\begin{cases} \lambda_1 = 2 > 0 \\ \lambda_2 = 2 > 0 \end{cases} \rightarrow \text{MINIMUM}$$

④ $f = -x^2 - y^2 - 1$

$$H\Big|_{x=0} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$(-2-\lambda)(-2-\lambda) = 0 \rightarrow \lambda_1 = -2 < 0$$
$$\lambda_2 = -2 < 0$$

② $f = x^2 - y^2 + 1$

$$H = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$\lambda_1 = 2 > 0$

$\lambda_2 = -2 < 0$

③ $f = x^2 + y^2 - xy + 1$

$$\frac{\partial f}{\partial x} = 2x - y$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial y \partial x} = -1$$

$$\det H = (2-\lambda)^2 - 1 = 4 + \lambda^2 - 4\lambda - 1 = \lambda^2 - 2\lambda + 3 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = \begin{cases} 3 \\ 1 \end{cases}$$

ok!

$\lambda_1 = 3 > 0$

$\lambda_2 = 1 > 0$

it is a MINIMUM!

≡

$$f(x,y) = x^2 + y^2 - 5xy$$

$$H = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$(2-\lambda)^2 - 25$$

$$\lambda^2 - 4\lambda - 21 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4(-21)}}{2} = \frac{4 \pm \sqrt{16 + 21 \cdot 4}}{2} \begin{cases} > 0 \\ < 0 \end{cases}$$

The present discussion can be done for each function $f(x,y)$ around the point (x_0, y_0) for which

$$\left(\vec{\nabla} f\right)_{\vec{x}=\vec{x}_0=(x_0,y_0)} = 0 .$$

$$f(x,y) \simeq f(x_0,y_0) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{\vec{x}_0} (x-x_0)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)_{\vec{x}_0} (y-y_0)^2$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\vec{x}_0} (x-x_0)(y-y_0) .$$

$$H_{\vec{x}_0} = \begin{pmatrix} \left(\frac{\partial^2 f}{\partial x^2}\right)_{\vec{x}_0} & \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\vec{x}_0} \\ \left(\frac{\partial^2 f}{\partial y \partial x}\right)_{\vec{x}_0} & \left(\frac{\partial^2 f}{\partial x^2}\right)_{\vec{x}_0} \end{pmatrix} .$$