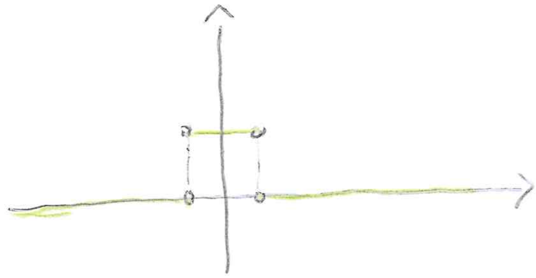
$$\delta_\varepsilon(x) = \begin{cases} 0 & |x| > \varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \end{cases}$$

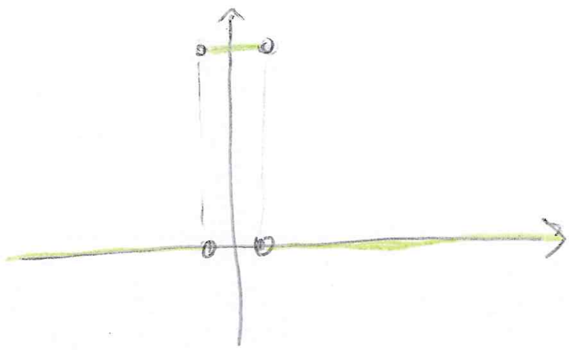
$\forall \varepsilon$ one has a different function.

$\varepsilon = \frac{1}{2}$

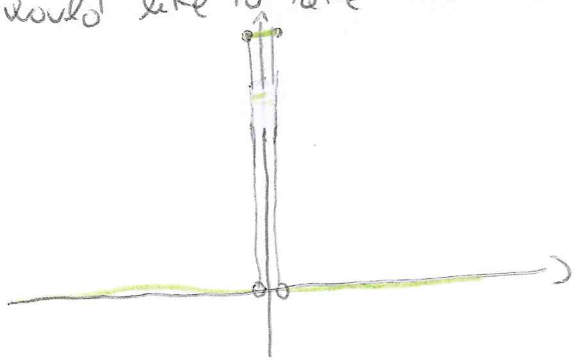


We are interested to small ε:

$\varepsilon = \frac{1}{10}$



In particular, we would like to take the limiting process $\varepsilon \mapsto 0$



$$\delta(x) \equiv \lim_{\varepsilon \mapsto 0^+} \delta_\varepsilon(x)$$

ACHTUNG:

the def. $\delta(x) = \lim_{\varepsilon \to 0^+} \delta_\varepsilon(x)$ is "pathological"...

what we get is not a function, but a rather strange construct which is zero everywhere and is ∞ for $x=0$...

Indeed, $\delta(x)$ is a "distribution" (generalisation of the concept of function).

The definition of the properties of distributions is not possible with the developed background.

However, even with an "intuitive approach" one can derive and describe almost all relevant features.

From a "physical" point of view:

The Dirac-δ appears everywhere: . mass and charge distribution of point-like particles.

- Fourier-Transform of "a pure sound": Dirac-delta in the frequency.
- Solution of so-called Green functions in E-Dyn and Field Theories.

$$\int_{-\infty}^{\infty} \delta(x)\,dx = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = 1$$

In fact: the function $\delta_\varepsilon(x)$ is constructed in such a way that

the area reads $A = 2\varepsilon \cdot \frac{1}{2\varepsilon} = 1.$

Formally:

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = \int_{-\infty}^{-\varepsilon} \delta_\varepsilon(x)\,dx + \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x)\,dx + \int_{\varepsilon}^{\infty} \delta_\varepsilon(x)\,dx =$$

$$= 0 + \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dx + 0 =$$

$$= \frac{1}{2\varepsilon} \cdot (x)\Big|_{-\varepsilon}^{+\varepsilon} = \frac{1}{2\varepsilon}\left(\varepsilon - (-\varepsilon)\right) = \frac{2\varepsilon}{2\varepsilon} = 1\,!$$

Note, the result is indipendent on ε. Ergo:

$$\boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1}$$

The 'function' $\delta(x)$ is such that is zero everywhere except for $x=0$, and for $x=0$ is $\delta(0) = \infty$ in such a way that the area is 1 :)

Let us now calculate

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx \qquad \text{where } f(x): R \mapsto R .$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = \int_{-\infty} f(x) \lim_{\varepsilon \mapsto 0} \delta_\varepsilon(x)\, dx = \lim_{\varepsilon \mapsto 0} \int_{-\infty}^{\infty} f(x)\, \delta_\varepsilon(x)\, dx =$$

$$= \lim_{\varepsilon \mapsto 0} \left[\int_{-\varepsilon}^{\varepsilon} f(x)\, \delta_\varepsilon(x)\, dx \right] = \lim_{\varepsilon \mapsto 0} f(0) \frac{1}{2\varepsilon} 2\varepsilon = \lim_{\varepsilon \mapsto 0} f(0) = f(0).$$

Ergo:

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$$

(one writes also $f(x)\, \delta(x) = f(0)\, \delta(x)$)

Examples:

$$\int_{-\infty}^{\infty} e^{x}\, \delta(x)\, dx = e^{0} = 1$$

$$\int_{-\infty}^{\infty} (x+2)^2\, \delta(x)\, dx = 2^2 = 4$$

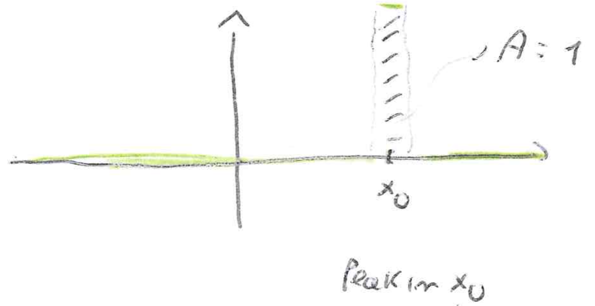
$$\int_{-\infty}^{\infty} x\, \delta(x)\, dx = 0$$

Note also that:

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = \int_{-a}^{a} \delta(x)\, f(x)\, dx \qquad \forall a > 0$$

We can shift the δ-function in a straitforward manner.

$\delta(x - x_0)$



The discussion is the same.

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0).$$

Examples:

$$\int_{-\infty}^{\infty} e^x \delta(x-1)\, dx = e$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \left[\delta(x-1) + \delta(x+1) \right] dx = \frac{1}{2} + \frac{1}{2} = 1$$

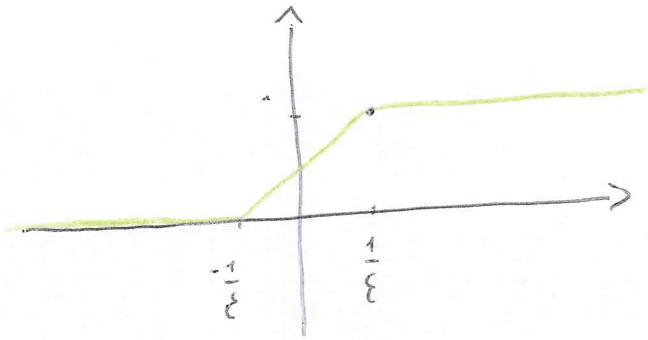
$$\int_{-\infty}^{\infty} x \left[\delta(x-3) - \delta(x+3) \right] dx = 3 - (-3) = 6.$$

Remind the function $\vartheta(x)$?

$$\vartheta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

$\vartheta(x)$ is strongly related to the $\delta(x)$. In order to see it let us express $\vartheta(x)$ also as a limiting process.

$$\vartheta_\varepsilon(x) = \begin{cases} 0 & x < -\varepsilon \\ \frac{1}{2\varepsilon}(x+\varepsilon) & |x| \leq \varepsilon \\ 1 & x > \varepsilon \end{cases}$$



Note that $\vartheta_\varepsilon(0) = \frac{1}{2}$ $\forall x$ (indipendently on ε!)

$$\vartheta(x) \equiv \lim_{\varepsilon \to 0} \vartheta_\varepsilon(x).$$

$$\frac{d\vartheta_\varepsilon}{dx} = \begin{cases} 0 & x \leq -\varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \\ 0 & x > \varepsilon \end{cases} = \begin{cases} 0 & |x| > \varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \end{cases} = \delta_\varepsilon(x)$$

Ergo we have that

$$\frac{d\vartheta_\varepsilon(x)}{dx} = \delta_\varepsilon(x)$$

By taking the limit $\varepsilon \mapsto 0$ we get

$$\boxed{\delta(x) = \frac{d\vartheta(x)}{dx}}$$

Conversely, one can also express $\vartheta(x)$ as

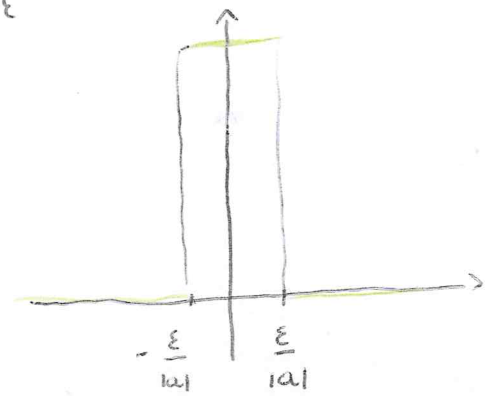
$$\vartheta(x) = \int_{-\infty}^{x} \delta(z)\,dz\,.$$

Important property

$$\boxed{\delta(ax) = \frac{1}{|a|}\,\delta(x) \qquad \forall a \neq 0.}$$

$$\delta(ax) = \lim_{\varepsilon \to 0} \delta_\varepsilon(ax)$$

$$\delta_\varepsilon(ax) = \begin{cases} 0 & \text{for } |ax| > \varepsilon \\ \frac{1}{2\varepsilon} & \text{for } |ax| < \varepsilon \end{cases} = \begin{cases} 0 & \text{for } |x| > \frac{\varepsilon}{|a|} \\ \frac{1}{2\varepsilon} & \text{for } |x| \leq \frac{\varepsilon}{|a|} \end{cases}$$



$$A = \frac{1}{2\varepsilon} \cdot \frac{2\varepsilon}{|a|} = \frac{1}{|a|}$$

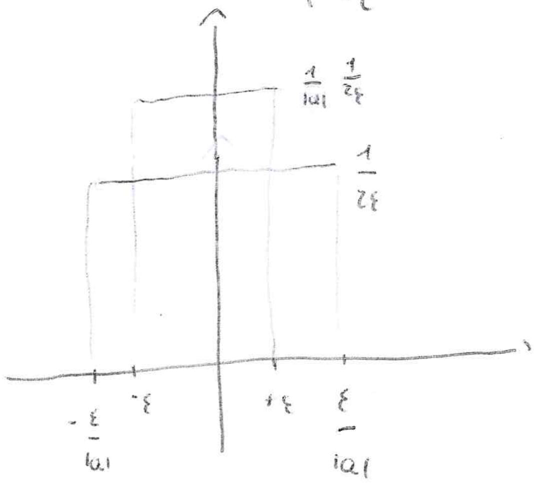
$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{|a|} \qquad \int_{-\infty}^{\infty} \delta(ax)\,f(x)\,dx = \frac{1}{|a|}\,f(0)$$

We therefore can write

$$\delta(ax) = \frac{1}{|a|}\,\delta(x).$$

Note that this is true when the limit is taken, but not for a finite ε.

$$\delta_\varepsilon(ax) \neq \frac{1}{|a|}\delta_\varepsilon(x) = \frac{1}{|a|} \cdot \begin{cases} 0 & |x| > \varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \end{cases}$$



same A, but different shape... but for $\varepsilon \mapsto 0$ you do not see any difference. This shows clearly the peculiarity of a limiting process.

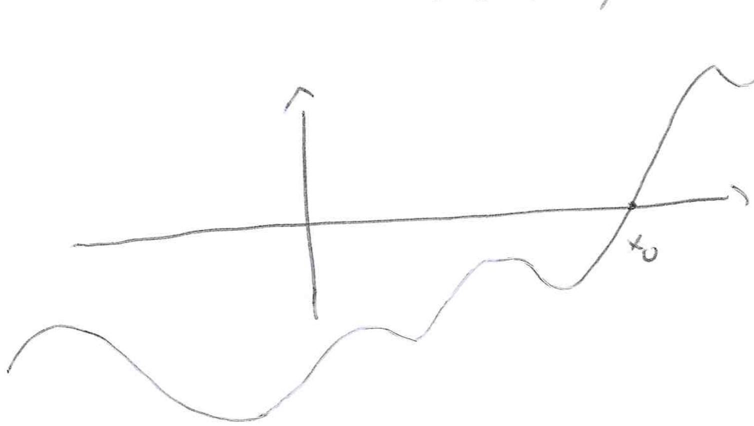
Simple generalization:

$$\boxed{\delta(a(x-x_0)) = \frac{1}{|a|}\delta(x-x_0)}$$

$f(x): \mathbb{R} \mapsto \mathbb{R}$

For x_0 / $f(x_0) = 0$. Let us assume that: x_0 is the only point for which $f(x)$ vanishes.

Moreover, we also assume that $f'(x_0) \neq 0$.



What is $\delta(f(x))$ in this case?

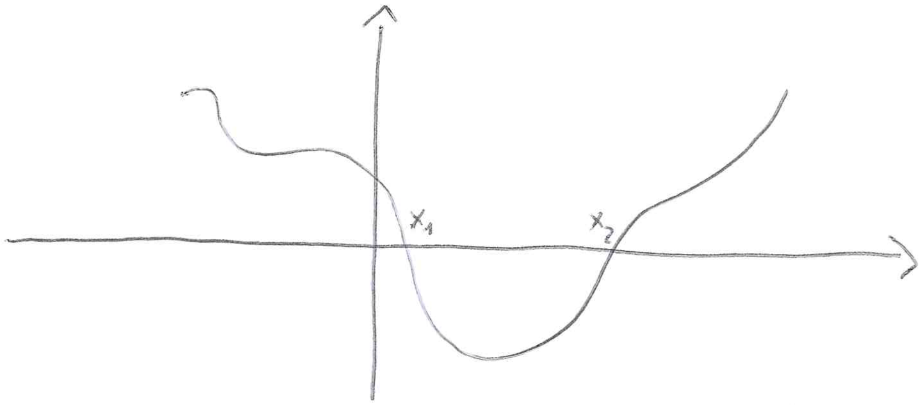
$\forall x \neq x_0$ $f(x) \neq 0$; ergo: $\delta(f(x)) = 0$ $\forall x \neq x_0$.

For $x = x_0$ we must be careful; for $x \simeq x_0$ we can approximate $f(x)$ as a Taylor expansion:

$$f(x) = \underbrace{f(x_0)}_{\substack{=0 \\ \text{per hp}}} + \underbrace{f'(x_0)}_{\substack{\neq 0 \\ \text{per hp}}} \cdot (x - x_0) + \dots = f'(x_0)(x - x_0)$$

Ergo:

$$\delta(f(x)) = \delta(f'(x_0)(x - x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$



if $f(x)$ / $f(x_1)=0$, $f(x_2)=0$ and $f'(x_1)\neq 0$, $f'(x_2)\neq 0$

we can repeat the same reasoning:

$$\delta(f(x))=0 \quad \forall x\neq x_1,\ x\neq x_2 .$$

For $x\simeq x_1$ write $f(x)\simeq f'(x_1)(x-x_1)$

$$\delta(f(x)) \underset{x\simeq x_1}{\simeq} \frac{1}{|f'(x_1)|}\,\delta(x-x_1)$$

For $x\simeq x_2$ write $f(x)\simeq f'(x_2)(x-x_2)$

$$\delta(f(x)) \underset{x\simeq x_2}{\simeq} \frac{1}{|f'(x_2)|}\,\delta(x-x_2)$$

Put alltogether:

$$\boxed{\delta(f(x))=\frac{1}{|f'(x_1)|}\,\delta(x-x_1)+\frac{1}{|f'(x_2)|}\,\delta(x-x_2)}$$

We can generalise this procedure:

$f(x): \mathbb{R} \mapsto \mathbb{R}$

with $\begin{cases} f(x_i) = 0 \\ \text{and} \\ f'(x_i) \neq 0 \end{cases}$ $(i = 1, \dots, N)$.

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{1}{|f'(x_i)|} \delta(x - x_i).$$

Example:

$\delta(x^2 - a^2)$ $(a > 0)$.

$f(x) = x^2 - a^2$; $f(x) = 0 \rightarrow x^2 - a^2 = 0$ $x = \pm a$. $\begin{pmatrix} x_1 = a \\ x_2 = -a \end{pmatrix}$.

$f'(x) = 2x$; $\begin{cases} f'(x_1) = 2a \neq 0 \\ f'(x_2) = -2a \neq 0 \end{cases}$

Ergo:

$$\delta(x^2 - a^2) = \frac{1}{|2a|} \delta(x - a) + \frac{1}{|-2a|} \delta(x + a) =$$

$$= \frac{1}{2a} \delta(x - a) + \frac{1}{2a} \delta(x + a).$$

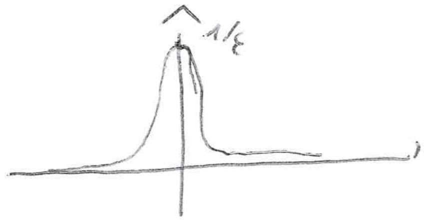
For instance:

$$\int_{-\infty}^{\infty} x^2 \delta(x^2 - a^2) dx = \frac{a^2}{2a} + \frac{1}{2a} (-a)^2 = a.$$

The chosen way to determine the $\delta(x)$ as a limit $\varepsilon \mapsto 0$ is <u>not</u> unique!

For instance, let us consider

$$L_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2+\varepsilon^2}$$



(For $\varepsilon \mapsto 0$ we get also in this case a very peaked function for $x=0$ and a vanishing function elsewhere).

$$\int_{-\infty}^{\infty} L_\varepsilon(x)dx = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2+\varepsilon^2} = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{dx}{\varepsilon^2\left(\frac{x^2}{\varepsilon^2}+1\right)} \qquad z=\frac{x}{\varepsilon} \qquad dx = \varepsilon dz$$

$$= \frac{\varepsilon}{\pi}\cdot\frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}\frac{\varepsilon\, dz}{z^2+1} = \frac{1}{\pi}\left[\arctan z\right]_{-\infty}^{\infty} = \frac{1}{\pi}\Big[\underbrace{\arctan \infty}_{\frac{\pi}{2}} - \underbrace{\arctan(-\infty)}_{\left(-\frac{\pi}{2}\right)}$$

$$= 1 \; ! \quad (\forall \varepsilon).$$

$$\boxed{\delta(x) = \lim_{\varepsilon \mapsto 0^+} L_\varepsilon(x)}$$

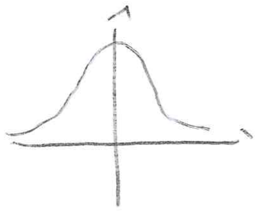
All the properties studied before are "still" fulfilled.

Indeed, there is an ∞ of ways to obtain $\delta(x)$. One speaks of different representations.

The employed one depends on the problem under study.

Another useful representation is the Gaussian:

$$G_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$$



For $\varepsilon \mapsto 0^+$ it becomes extremely peaked.

$$\int_{-\infty}^{\infty} G_\varepsilon(x)\, dx = 1 \quad \forall \varepsilon.$$

$$\boxed{\lim_{\varepsilon \mapsto 0} G_\varepsilon(x) \equiv \delta(x)}$$