Digression: Countability



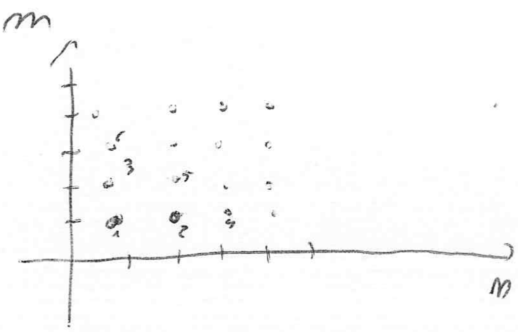
Are there more "natural numbers" or "even numbers"?

Indeed, they are the same. In fact, I can find a biunivocal correspondence among them.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \dots \\ \updownarrow & \updownarrow & \updownarrow & & & \\ 2 & 4 & 6 & 8 & 10 & \dots \end{array}$$

What about the numbers

$\dfrac{n}{m}$ $n = 1, 2, \dots$
$m = 1, 2, \dots$



1 2 3 4 5 6 7 ...

(1,1) (2,1) (1,2) (3,1) ...

1/1 2/1 1/2 ....

However, not all the numbers can be written as $\frac{n}{m}$ ....

Example:

$\sqrt{2} = \frac{n}{m}$ whereas $n, m$ are not both even (otherwise $n = 2p$, $m = 2q$ and get $p/q$ ...

$2 = \frac{n^2}{m^2}$ $\quad n^2 = 2m^2 \to n^2$ even $\to n$ even ...

But if $n$ is even: $n = 2p$

$4p^2 = 2m^2 \to m^2 = 2p^2 \to m^2$ even $\to m$ even ...

But then we have obtained that both $(n, m)$ are even ...

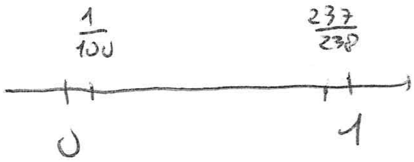
Contrary to our hypothesis.

That is, we cannot write $\sqrt{2}$ as a rational number.

[Pythagoras's story ...] $\quad \sqrt{2}$ is irrationals ...

Real numbers: rational + irrational.

Indeed, how many real numbers exist?




The "rational nos" form a dense system... but in there there are also real numbers, such as $1/\sqrt{2}$, $1/\pi$ ...

The rational numbers are countable, as we have seen.

But what about real numbers? Let us assume that it is so:

1) $0.12578\ldots$

2) $0.42134\ldots$

3) $0.91241\ldots$

4) $0.91783\ldots$

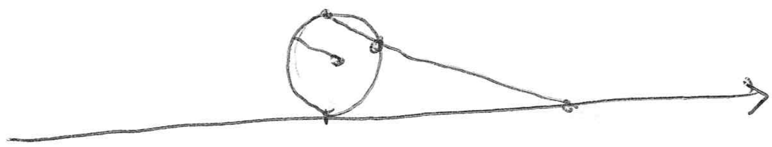
...

We now construct the following number "x" $\in (0,1)$:

$x = 0.2339\ldots$

Then we get a number which is "per construction" different from each number of the list. We have thus shown that the real numbers are <u>not</u> countable.

But even more than that.. Between a finite segment we have all the points of an infinite line

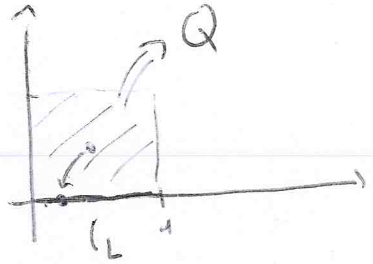


But more on that: one can also construct biunivocal relation between the full plane and a segment on the line.

⇝ Peano curve

Is there something between "countable and uncountable"? open issue...

Another "clever way" of writing down a connection between the plane and the line can be done as follows:

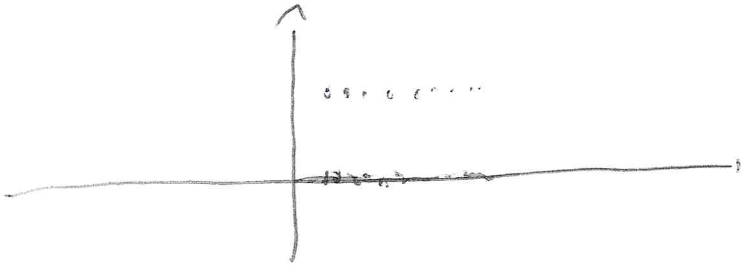



$P = (x, y) \equiv (0,1278\ldots, 0.0591\ldots) \Longleftrightarrow 0.10257981\ldots$

Univocal "relation" between Q and L.

The Dirichlet function:

$$d(x) = \begin{cases} 1 & \text{for } x = n/m \\ 0 & \text{for } x \text{ irrational} \end{cases}$$



$d(x)$ is 'nowhere' continuous...

However, there are "more" irrational nos than "rational" ones...

The correct expression is that the set of rational numbers is "dense" but has "zero measure"...

$$\int_a^b d(x)\,dx = 0 \quad !$$