Let us consider $f(x): \mathbb{R} \mapsto \mathbb{R}$ with a Taylor-expansion:

$$f(x) = a_0 + a_1 x + \frac{a_2}{2} x^2 \ldots$$

Let us now study the replacement $x \mapsto z = x + iy$

$$f(z)$$

$$f(z) = a_0 + a_1 z + \frac{a_2}{2} z^2 + \ldots$$

(in general, we can also consider the constant a_i as complex, but that is not important now).

Example:

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,(2xy).$$

So, summa summarum:

$$f(z) : \mathbb{C} \mapsto \mathbb{C}$$

is a function which depends on z.

it is an "analytic function"...

Achtung:

$f(z) = z^* = x - iy$ is not analytic..., The Derivative can be easily built:

$$\frac{d f(z)}{dz} = \lim_{h \mapsto 0} \frac{f(z+h) - f(z)}{h}$$

This is a highly non-trivial fact... In $R^2 \mapsto R^2$ we actually saw that there is no well defined limit...

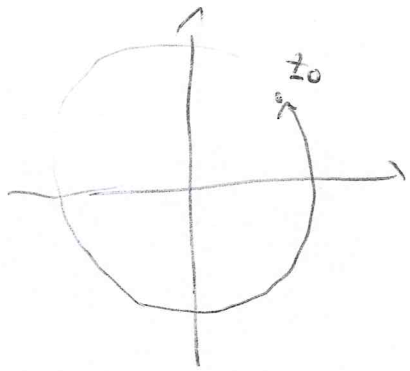
$$\frac{d z^n}{dz} = n z^{n-1}$$

On a practical level, everything as "before"....

=

Generally:

$$f(z_0) = f(z_0 \cdot e^{i 2\pi})$$



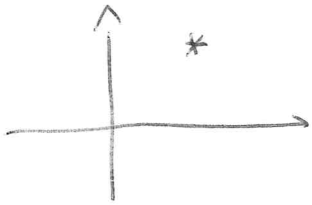
This is true for well "behaved" function...

Already at this level we can make some examples:

(*) $f(z) = \dfrac{1}{z}$

Polo in "0"

$f(z) = \dfrac{1}{z - z_0}$



"Pole in z_0"

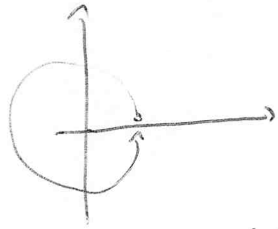
(*) $f(z) = \sqrt{z}$

$z = \rho e^{i\varphi}$

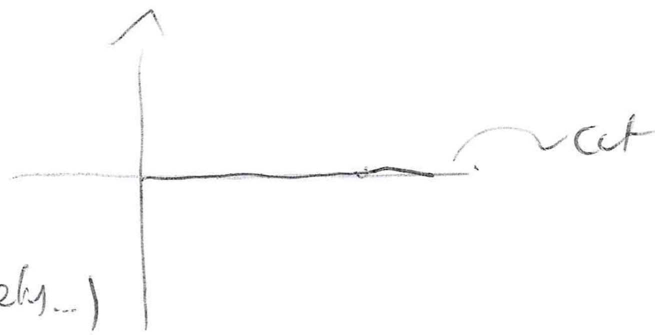
$f(z) = \sqrt{\rho}\; e^{i\varphi/2}$

$\varphi = 0 \qquad f(z) = \sqrt{\rho}$

$\varphi = 2\pi \qquad f(z) = -\sqrt{\rho}$



In order to go 'back' you need to turn twice!!!!!!

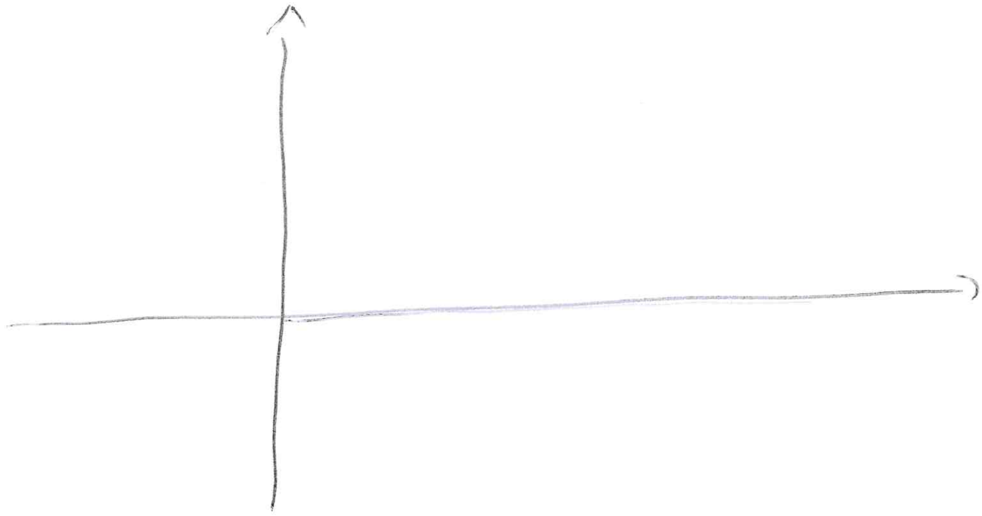


Riemann sheets ... (2 sheets...)

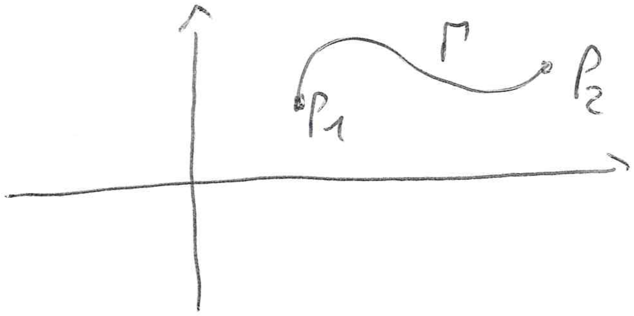
$f : R \mapsto \mathbb{C}$

$f(z) = \ln z$

$$\ln(\varrho e^{i\varphi} \cdot \underbrace{e^{i2\pi n}}_{1}) = \ln \varrho + i(\varphi + 2\pi n)$$



∞ nr of sheets...



What is

$$I = \int_{P_1, \Gamma}^{P_2} f(z)\, dz \quad ?$$

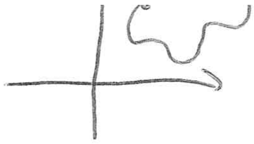
Now, we can find $z = z(t)$ / $z(t_1) = P_1$ and $z(t_2) = P_2$

$$I = \int_{t_1}^{t_2} f(z(t)) \frac{dz(t)}{dt}\, dt$$

Now, consider $F(z)$ / $\dfrac{dF(z)}{dz} = f(z)$.

We then have:

$$I = \int_{t_1}^{t_2} \frac{dF(z)}{dz} \frac{dz}{dt}\, dt = \int_{t_1}^{t_2} \frac{d\, F(z(t))}{dt}\, dt = F(z(t_2)) - F(z(t_1))$$
$$= F(P_2) - F(P_1).$$

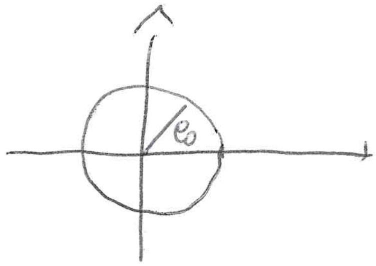


Ergo:

$$\oint f(z)\,dz = 0 \;!!!!\quad \text{if } f(z) \text{ is analytic...}$$

whatever closed curve

Explicit ex.



$$\oint_C z^2\,dz$$

$$z = \rho_0 e^{i\varphi}$$

$$dz = \rho_0\, i\, d\varphi\, e^{i\varphi}$$

Ergo

$$\oint_C z^2\,dz = \int_0^{2\pi} \rho_0^2 e^{i2\varphi}\, \rho_0\, i\, d\varphi\, e^{i\varphi} =$$

$$= \rho_0^3\, i \int_0^{2\pi} e^{i3\varphi}\, d\varphi = i \rho_0^3 \frac{1}{i3} \left(e^{i3\varphi} \right)_0^{2\pi} = 0 \;!!!$$

Now, let us consider a "peculiar case" ....

$$f(z) = \frac{c}{z}$$

$$\oint_c \frac{c}{z} dz$$

$$z = \rho_0 e^{i\varphi} \qquad \varphi \in (0, 2\pi).$$

$$\oint_c \frac{c}{z} dz = \int_0^{2\pi} \frac{c}{\rho_0 e^{i\varphi}} \rho_0 i e^{i\varphi} d\varphi = ic \int_0^{2\pi} d\varphi = ic2\pi =$$

$$= 2\pi i c$$

In general, one defines:

$$\boxed{\operatorname{Res}\left[f(z)\right]_{z=z_0} = \frac{1}{2\pi i}\oint_\gamma f(z) dz}$$

$$\operatorname{Res}\left[\frac{c}{z}\right]_{z=0} = \frac{1}{2\pi i}\oint_\gamma f(z) dz = c \; !!!!$$

Note that

$$\oint z^m \, dz = 0 \quad \text{for each } m = 0, 1, 2, \ldots \quad -2, -3, -4$$

but not for $m = -1$.

In general a function $f(z)$ can be expanded in Taylor if it has no poles.

For $z = 0$

$$f(z) = \ldots \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$

$$\oint f(z) \, dz = 2\pi i \, (a_{-1})$$

For $z \simeq z_0$

$$f(z) = \ldots \frac{a_{-2}}{z - z_0} + \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 \ldots$$