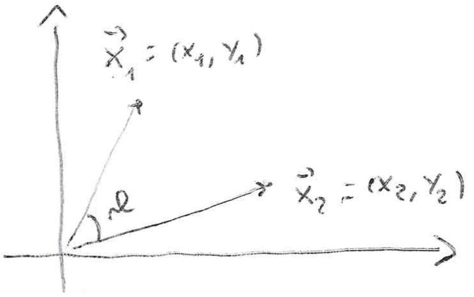
Scalar product

Let us consider the plane $\mathbb{R}^2$.



We recall that the lengths are:

$$\ell(\vec{x}_1) = \|\vec{x}_1\| = \sqrt{x_1^2 + y_1^2}$$

$$\ell(\vec{x}_2) = \|\vec{x}_2\| = \sqrt{x_2^2 + y_2^2}$$

The scalar product $\vec{x}_1 \cdot \vec{x}_2$ is defined as

$$\vec{x}_1 \cdot \vec{x}_2 = x_1 x_2 + y_1 y_2$$

It is indeed possible to write the scalar product as

$$\vec{x}_1 \cdot \vec{x}_2 = \|\vec{x}_1\| \cdot \|\vec{x}_2\| \cdot \cos\alpha$$

where α is the angle between $\vec{x}_1$ and $\vec{x}_2$.

The proof is an exercise of "Sheet 6".

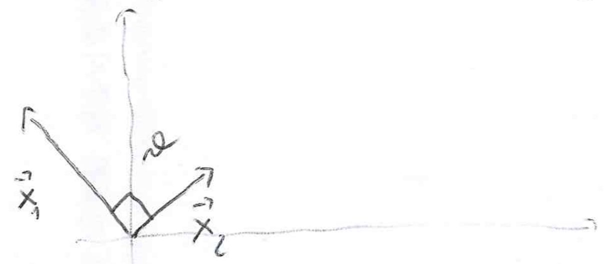
Note: gener. to N dimensions is trivial:

$\vec{x} = (x_1, \dots, x_N)$ $\qquad \vec{x} \cdot \vec{y} = \sum_{i=1}^{N} x_i y_i = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos\alpha$

$\vec{y} = (y_1, \dots, y_N)$

A very simple consequence of the scalar product is that:

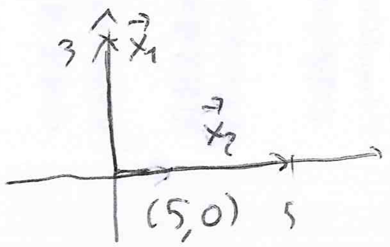
$\vec{x}_1 \cdot \vec{x}_2 = 0 \quad \text{if } \vartheta = \frac{\pi}{2}$



$\vec{x}_1 \cdot \vec{x}_2 = 0$.

(this is so even if $\|\vec{x}_1\| \neq 0$ and $\|\vec{x}_2\| \neq 0$).

Example:

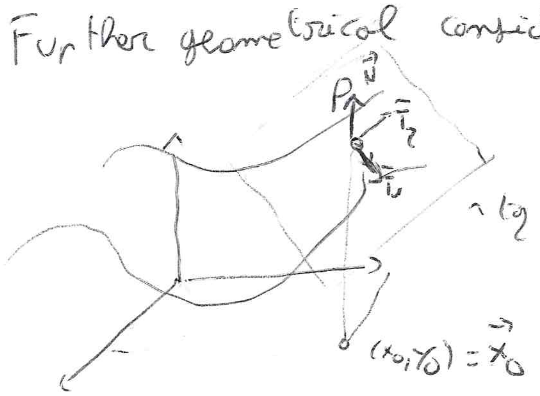


$\vec{x}_1 = (0,3)$

$\vec{x}_2 = (5,0)$

$\vec{x}_1 \cdot \vec{x}_2 = (0,3)\cdot(5,0) = 0\cdot 5 + 3 \cdot 0 = 0$ q.e.d.

Further geometrical consideration (1) :



$f(x,y): \mathbb{R}^2 \mapsto \mathbb{R}$

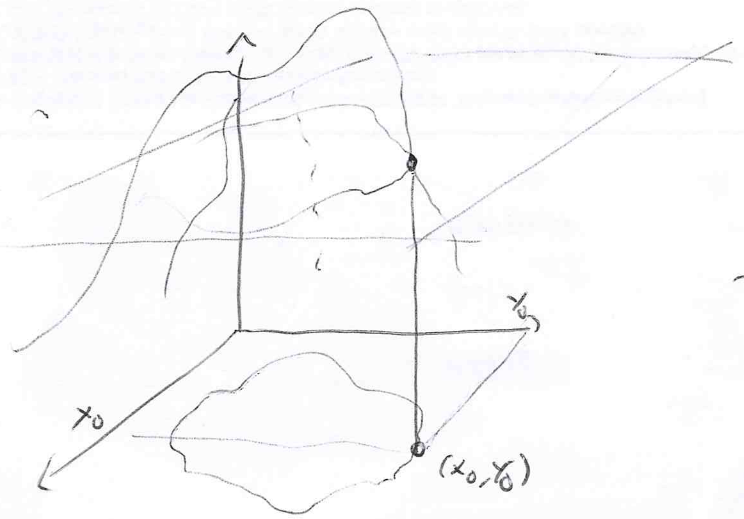
$$\vec{T}_1 = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2_{\vec{x}_0}}} \left(1, 0, \left(\frac{\partial f}{\partial x}\right)_{\vec{x}_0}\right)$$

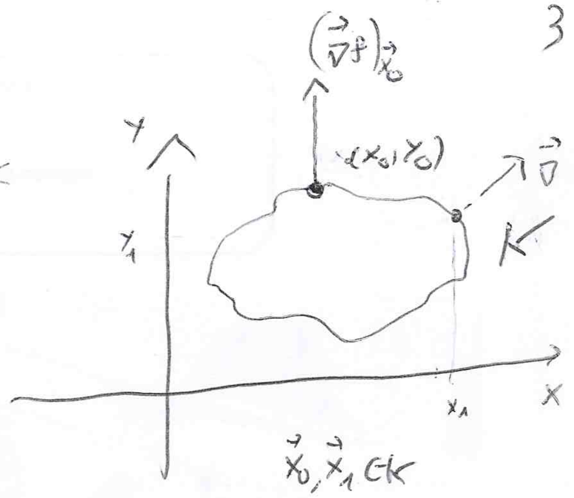
$$\vec{T}_2 = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial y}\right)^2_{\vec{x}_0}}} \left(0, 1, \left(\frac{\partial f}{\partial y}\right)_{\vec{x}_0}\right)$$

Now, there is also a "normal vector" $\vec{N}$ which is orthogonal ($\perp$) to $\vec{T}_1$ and $\vec{T}_2$ and has length 1.

The exercise 1.2 is the following: determine the general expression of $\vec{N}$.

Further geometrical considerations (2)





$f(x,y): \mathbb{R}^2 \mapsto \mathbb{R}$

Let us consider $\vec{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ and $z_0 = f(x_0, y_0)$.

Let us then define the curve K in $\mathbb{R}^2$ as:

$$K: \{(x,y) \text{ such that } f(x,y) = z_0 = f(x_0, y_0)\}$$

Obviously, $(x_0, y_0) \in K$.

Now, it is possible to show that the gradient-vector

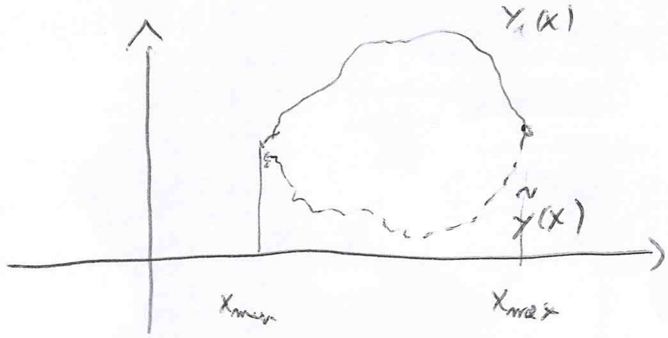
$$* \quad \left(\vec{\nabla} f\right)_{\vec{x}_0} = \left(\left(\frac{\partial f}{\partial x}\right)_{\vec{x}_0}, \left(\frac{\partial f}{\partial y}\right)_{\vec{x}_0}\right) \text{ is \underline{orthogonal} to the curve } K.$$

The proof is the Ex. 1.3 of sheet 6.

Hint: in the vicinity of $\vec{x}_0$ the curve K can be expressed as $(x, y(x))$, where $y(x) \,/\, f(x, y(x)) = z_0$.

* $\vec{\nabla} f$ is $\perp$ to K for each point $(x,y) \in K$. For instance, consider $\vec{x}_1 \in K$: $\vec{\nabla} f \perp K$ in $\vec{x}_1$ as well.

For instance, in the present graphical example:



$$K = \left\{ (x, y(x)) \text{ and } (x, \tilde{y}(x)) \quad \text{for } x \in [x_{min}, x_{max}] \right\}$$

Indeed, it is always possible to find $y(x)$ which fulfills

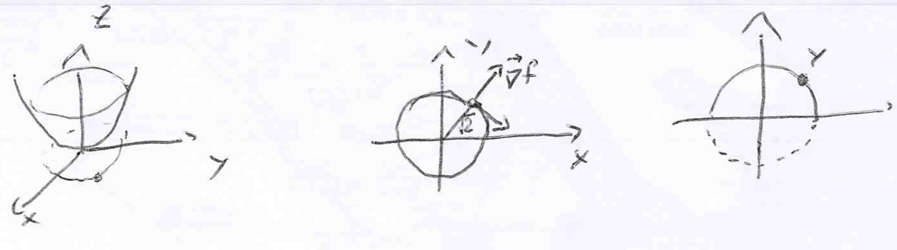
$$f(x, y(x)) = z_0$$

in the neighbourhood of (x_0, y_0).

This is the "so called" Dini's Theorem (Satz von der impliziten Funktion)

(Implicit function theorem)

<u>Simple example</u>



$f(x,y) = x^2 + y^2$

$(x_0, y_0) = (1, 1)$

$z_0 = f(x_0, y_0) = x_0^2 + y_0^2 = 2$

$x^2 + y^2 = 2$

$y = \sqrt{2 - x^2}$

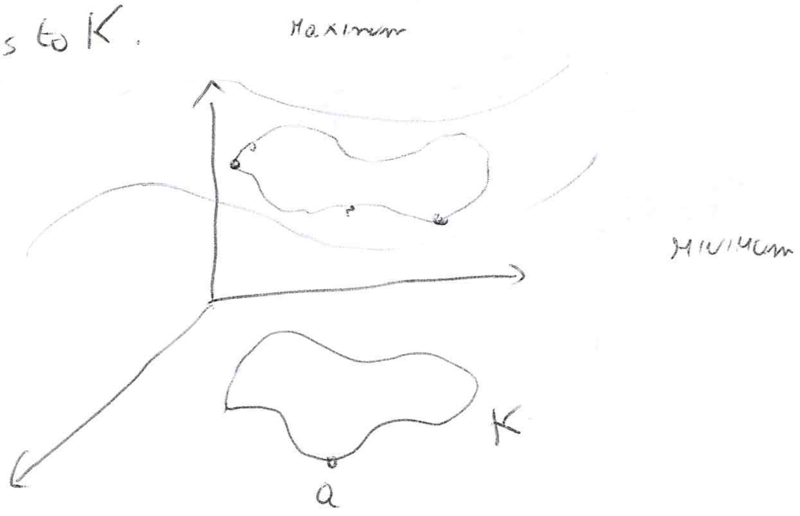
$\tilde{y} = -\sqrt{2 - x^2}$

Note:

$\vec{\nabla} f = (2x, 2y)$ $\quad \left.\vec{\nabla} f\right|_{\vec{x}_0} = (2, 2)$.

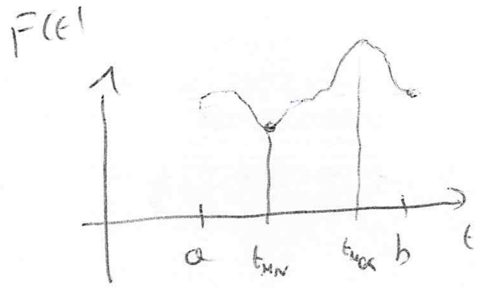
The verification that $(\vec{\nabla} f)_{\vec{x}_0}$ is $\perp$ to the circle is left to you.

Suppose now to have a curve K on the (x,y) plane and search for extrema (maxima or minima) on f(x,y) whereas (x,y) belongs to K.



How to find maxima and minima bound to the constrain?

$$K : \{ (x(t), y(t)) , t \in (a,b) \}$$



Then, consider

$$F(t) = f(x(t), y(t))$$

$F(a) = F(b)$

Then in 3d

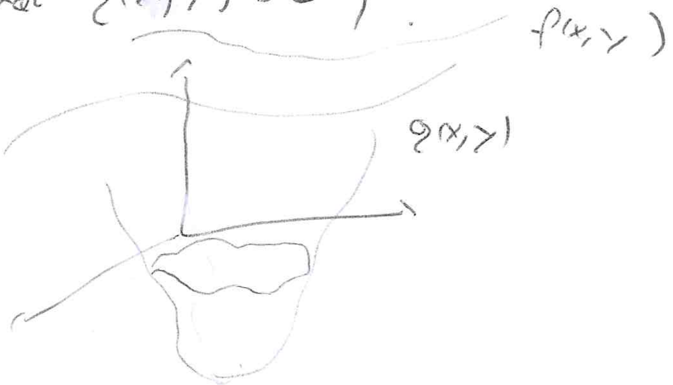
and Search for the points

$$\frac{dF(t)}{dt} = 0$$

$$\frac{dF(t)}{dt} = \left(\frac{\partial f}{\partial x}\right)_{\vec{x}(t)} \frac{dx(t)}{dt} + \left(\frac{\partial f}{\partial y}\right)_{\vec{x}(t)} \frac{dy}{dt}$$

Now, let us consider the case in which

$K : \{ (x,y) \text{ such that } g(x,y) = 0 \}$.



Of course, we could search for $(x(t), y(t))$ such that $g(x(t), y(t)) = 0$

and we are back to the previous form.

But there is a more elegant and general trick, called "the method of the Lagrange multiplier":

Consider the function

$$h(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

Now study the extreme for $h(x,y,\lambda)$.

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0$$

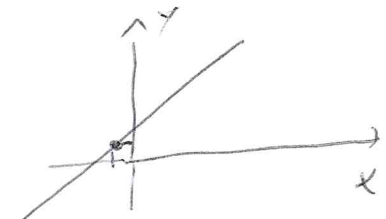
$$\frac{\partial h}{\partial \lambda} = g(x,y) = 0 \longrightarrow$$ the constrain is ensured...

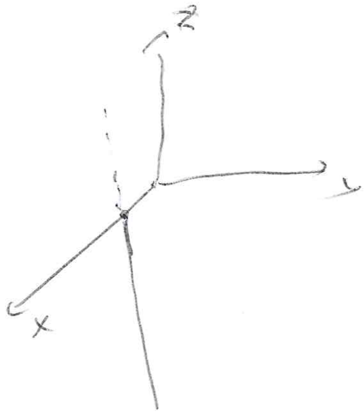
A solution is a point (x_0, y_0, λ_0) ...

Example:

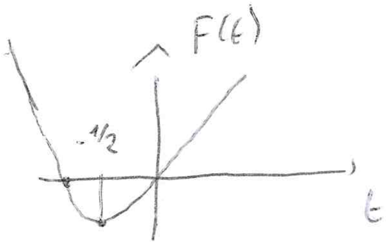
$$f(x,y) = xy$$

$$K : \{ \ (x(t) = t, y(t) = t+1\} \quad t \in (-\infty, \infty) \}$$





Method 1:



$$F(t) = f(t, t+1) = t \cdot (t+1) = t^2 + t$$

$$\frac{dF}{dt} = 2t + 1 = 0 \rightarrow t = -\frac{1}{2}$$

Ergo

$$(t, t+1) = \left(-\frac{1}{2}, \frac{1}{2}\right) \Rightarrow$$ this corresponds to an extremum...

Indeed, it is a minimum, because $\frac{d^2F}{dt^2} = 2 > 0$

$$f(x,y) = \left(-\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right) = -\frac{1}{4}$$

Note that:

$$\frac{dF}{dt} = \left(\frac{\partial f}{\partial x}\right)_{\vec{x}(t)} \frac{dx(t)}{dt} + \left(\frac{\partial f}{\partial y}\right)_{\vec{x}(t)} \frac{dy(t)}{dt} = (y)_{\vec{x}(t)} \cdot 1 + (x)_{\vec{x}(t)} \cdot 1 =$$

$$= (t+1) + t = 2t + 1$$

q.e.d.

Let us repeat the ex. with the method of Lagrange multiplier:

$$K = \{(x,y) / \underbrace{y - x - 1 = 0}_{g(x,y)=0}\}$$

$$h(x,y,\lambda) = f(x,y) - \lambda g(x,y) = xy - \lambda(y - x - 1).$$

$$\partial_x h = y - \lambda(-1) = 0 \qquad y = -\lambda$$

$$\partial_y h = x - \lambda = 0 \qquad x = \lambda$$

$$\partial_\lambda h = y - x - 1 = 0 \quad \rightarrow \quad y = x + 1$$

solve the system: from (1) - (2)

$$y + x = 0 \rightarrow y = -x$$

put in (-3)

$$-x = x + 1 \rightarrow 2x = -1 \qquad x = -\frac{1}{2} \quad \Rightarrow \quad y = -x = +\frac{1}{2}$$

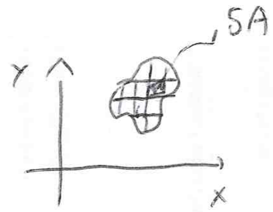
$\left(-\frac{1}{2}, \frac{1}{2}\right)$ is the point on the (x,y)-plane which correspond to an extremum (in this case a minimum) for the "constrained system". Note: same solution.

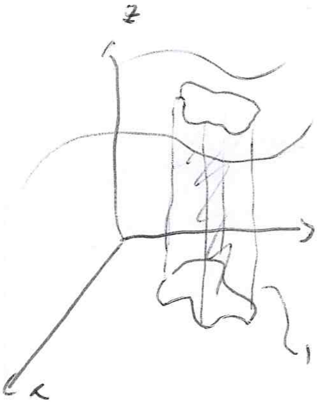
Note, in this case it is more difficult to say if we have a maximum or a minimum.

Integral in 2D = Volume

$f(x,y)$

$R \subseteq \mathbb{R}^2$





How to calculate the volume V between the (x,y)-plane and the function f(x,y)?

Divide R into N (equal) small areas δA...

$$V \cong \sum_{i=1}^{N} f(\vec{x}_i)\,\delta A$$

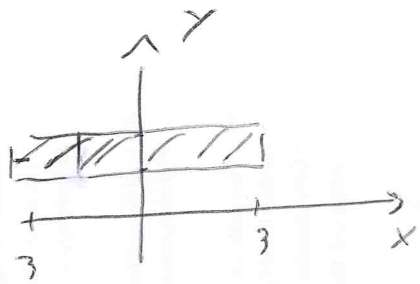
Obviously, the smaller δA, the "more precise" is the evaluation of the volume.

We take the limit $\delta A \mapsto 0$. $\delta A = dx\,dy$ (small rectangular)

$$V = \lim_{\delta A \mapsto 0} \sum_{i=1}^{N} f(\vec{x}_i)\,\delta A = \int_R dx\,dy\, f(x,y)$$

Examples

1) $R: \{ -3 \leq x \leq 3 \ , \ 1 \leq y \leq 2 \}$



$f(x,y) = x + 2y$

$$\int_R dx\, dy\ (x+2y) = \int_{-3}^{3} dx \int_{1}^{2} (x+2y)\, dy = \int_{-3}^{3} dx \left[x \cdot y + 2\frac{y^2}{2} \right]_1^2 =$$

$$= \int_{-3}^{3} dx \left[2x + 4 - x - 1 \right] =$$

$$= \int_{-3}^{3} dx\ (x+3) = \left[\frac{x^2}{2} + 3x \right]_{-3}^{3} = \left[\frac{9}{2} + 9 - \frac{9}{2} + 3(-3) \right] =$$

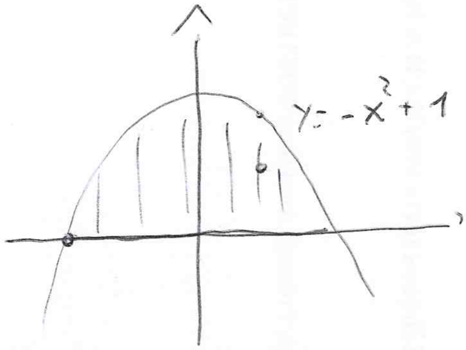
$$= 18$$

$V = 18 \quad [m^3].$

$R = \{(x,y) / 0 \le y \le -x^2+1\}$

$f(x,y) = x^2 y$

$$V = \int_R dx\,dy\, f(x,y) = \int_{-1}^{1} dx \int_{0}^{-x^2+1} (x^2 y)\, dy =$$



$-x^2+1=0$

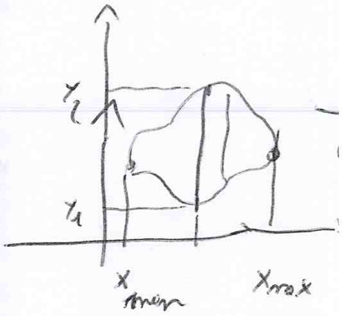
$x = \pm 1$

$$= \int_{-1}^{1} dx \left[x^2 \frac{y^2}{2} \right]_{y=0}^{y=-x^2+1} = \int_{-1}^{1} x^2 \frac{(-x^2+1)^2}{2}\, dx =$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2 - x^4)\, dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}\right] =$$

$$\frac{1}{2}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{1}{2}\,\frac{10-6}{15} = \frac{2}{15} = \checkmark$$

In general: $R: \{(x,y) / y_1(x) \le y \le y_2(x)\}$



$$\int_R f(x,y)\,dx\,dy = \int_{x_{min}}^{x_{max}} dx \int_{y_1(x)}^{y_2(x)} dy\, f(x,y)$$

Note, one can "reverse the ordering and perform first the integral over x and later over y ...

Suppose that we now want to calculate the integral

$$I = \int_R e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$

$$R: \{-\infty < x < \infty,\ -\infty < y < \infty\}$$

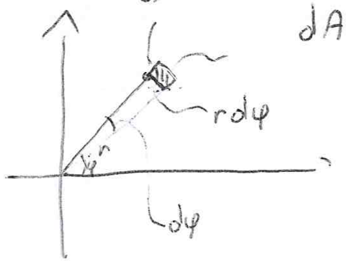
$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

But I have a problem... I cannot solve it. It is one of those integrals which one cannot solve...

A change of variable is in this case a "clever idea"...

Indeed, polar coordinates are well indicated for this...

However, care is needed when a change of variable is performed in a integral in more than one dimension.



$dA = dx\,dy = \underline{r\,dr\,d\varphi}$

(For small areas the different orienta is not important...

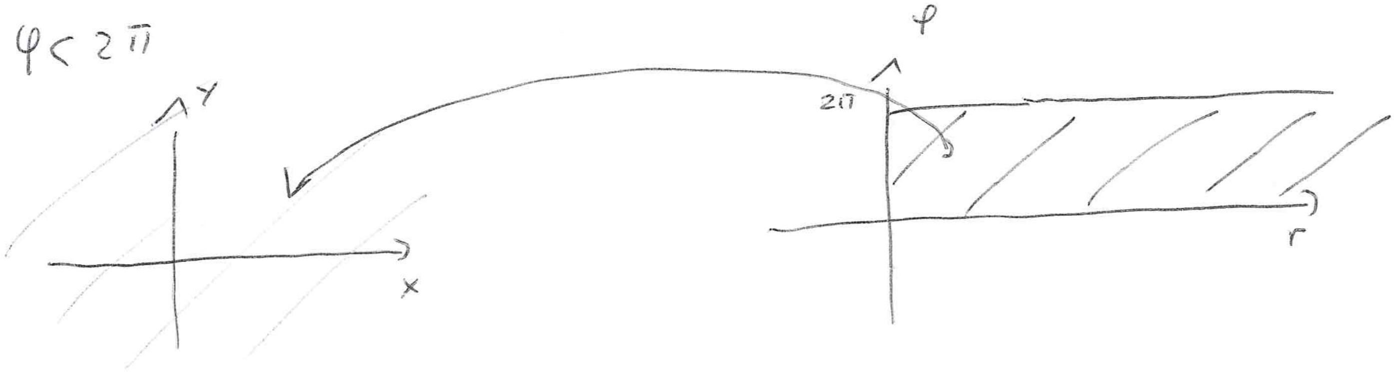
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\left(\frac{y}{x}\right) \end{cases}$$

Note also that:

$0 \leq r < \infty$

$0 \leq \varphi < 2\pi$



we have a univocal correspondence.

$$I = \underbrace{\int_0^{2\pi} d\varphi}_{} \int_0^{\infty} r\,dr\, e^{-\frac{1}{2}r^2}$$

$$= 2\pi \int_0^{\infty} r\,dr\, e^{-\frac{1}{2}r^2} = 2\pi \int_0^{\infty} e^{-w}\, dw = 2\pi \left[-e^{-w}\right]_0^{\infty} =$$

$$= 2\pi\left[0-(-1)\right]$$

$w = \frac{1}{2}r^2 \quad dw = r\,dr$

$$= 2\pi$$

$I = 2\pi.$

Note also that

$I = J^2$ with $J = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx.$

Therefore:

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}}$$

In general, if we have the transformation of coordinates

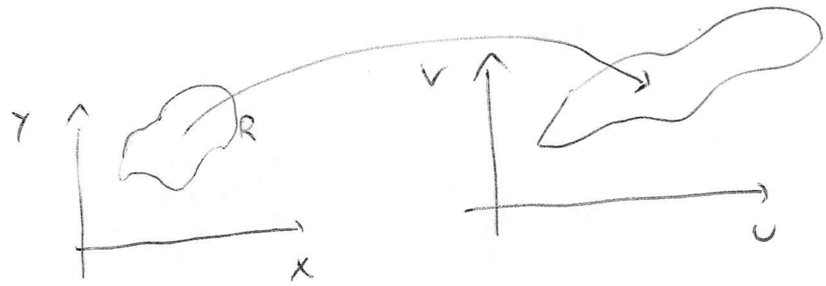
$(x,y) \mapsto (u,v)$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

the following formula holds:

$$\int_R dx\,dy\, f(x,y) = \int_{R'} du\,dv\, f(x(u,v), y(u,v))\, J$$

whereas



$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

For instance, for $(u,v) = (r,\varphi)$:

$x = r\cos\varphi$

$y = r\sin\varphi$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$dx\,dy \mapsto r\,dr\,d\varphi$.

Note, this is valid also in more dimensions:

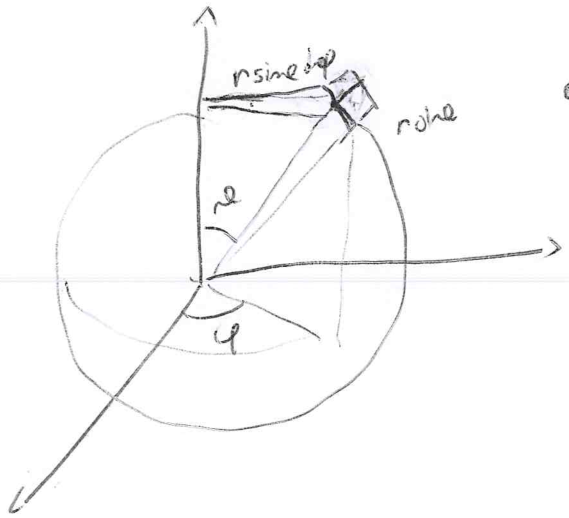
$$\int_R dx_1 \dots dx_N \, f(x_1, \dots, x_N)$$

$$x_i = x_i(y_1, \dots, y_N)$$

$$dx_1 \dots dx_N \to J \, dy_1 \dots dy_N$$

where $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_N} \\ \dots & & \\ \frac{\partial x_N}{\partial y_1} & & \frac{\partial x_N}{\partial y_N} \end{vmatrix}$

It is easy to prove that in spherical 3D coordinates one has



$dV = dr \, r d\theta \, r \sin\theta \, d\varphi =$

$= r^2 \sin\theta \, dr \, d\theta \, d\varphi$

The check with the