Limit

(Limes (Grenzwert))

$f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$ wobei D: Teilmenge von $\mathbb{R}$

$$\lim_{x \mapsto x_0} f(x) = L$$

"$f(x)$ hat für x gegen x_0 den Limes L,
wenn es zu jedem (noch so kleinen) $\varepsilon > 0$
ein (im Allgemeinen von ε abhängiges) $\delta > 0$ gibt,
sodass $\forall x \in D$ die
die Bedingung $|x - x_0| < \delta$ genügen,
auch $|f(x) - L| < \varepsilon$ gilt."

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

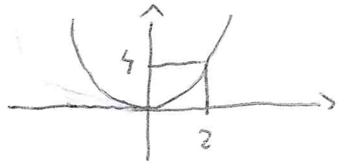
*) The concept of limit is crucial in mathematics, being one of the basic concepts on which all the rest is built.

*) This is a formalisation of a very intuitive idea: what happens if a variable tends to a particular value.

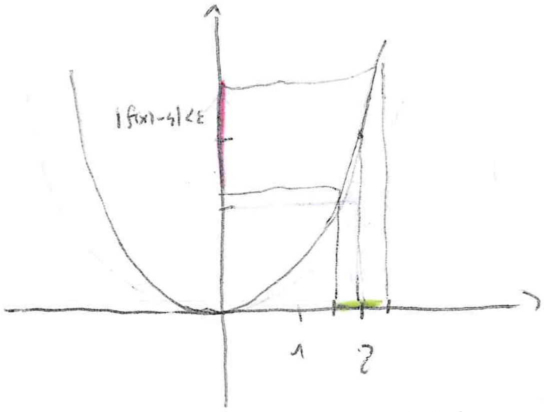
*) Generalisation of the definition for $x \mapsto \pm\infty$ and $L \mapsto \pm\infty$.

Example 1:

$$\lim_{x \mapsto 2} x^2 = 4$$



"actually trivial ..." but let us show that the definition works.



intuitive correspondence

if $|x-2|<\delta \rightarrow |x^2-4| = |(x-2)(x+2)| = |x-2||x+2| < \delta|x+2| = \delta|x-2+4|$

$$< \delta\big(|x-2|+4\big) < \delta(\delta+4) = \delta^2+4\delta = \varepsilon$$

$|x^2-4| < \delta^2+4\delta = \varepsilon \quad \rightarrow \quad \delta^2+4\delta-\varepsilon = 0$

Ergo: $\delta = -2 \pm \sqrt{4+\varepsilon} \quad \rightarrow \quad \delta \doteq \delta(\varepsilon) = -2+\sqrt{4+\varepsilon} \quad \rightarrow$ Not so "easy" indeed ...

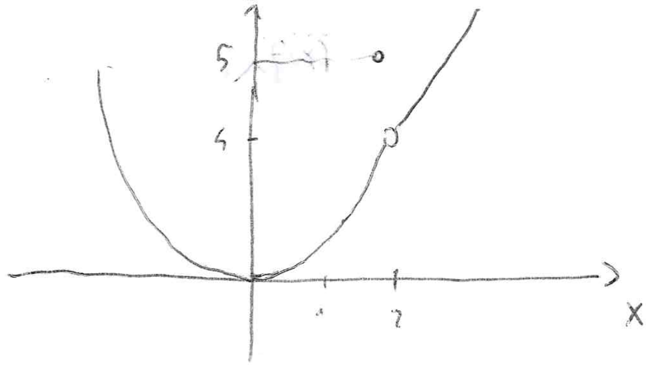
Summarizing, when closing

$$|x-2| < \underbrace{-2+\sqrt{4+\varepsilon}}_{\delta} \implies |x^2-4| < \varepsilon .$$

Example 2:

Let us define the function

$$f(x) = \begin{cases} x^2 & \text{for } x \neq 2 \\ 5 & \text{for } x = 2 \end{cases}$$



What is $\lim_{x \mapsto 2} f(x)$ in this case?

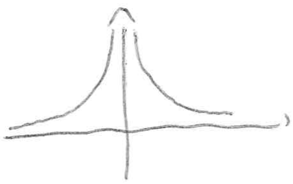
The limit is still 4! $\lim_{x \mapsto 2} f(x) = 4$. In fact, the limit is <u>not</u> always the value of the function in that point, but the value of the function when x is "tending" to that point.

Example 3:

$f(x) = \frac{1}{x^2} : D \subseteq \mathbb{R} \mapsto \mathbb{R}$

$D = (-\infty, 0) \cup (0, \infty)$

($x_0 = 0$ is not part of D)

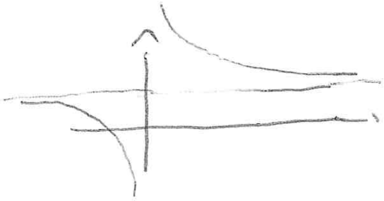


$$\lim_{x \mapsto 0} f(x) = +\infty$$

$\lim_{x \mapsto x_0} f(x) = +\infty$

$\forall N$ (very large) > 0 $\exists\, \delta > 0$ / $0 < |x - x_0| < \delta \rightarrow f(x) > N$

Example 4:



$$f(x) = \frac{1}{x} + 1$$

$$\lim_{x \mapsto +\infty} f(x) = 1$$

In general: $\lim_{x \mapsto +\infty} f(x) = L$

$$\forall \varepsilon > 0 \quad \exists N > 0 \; / \; x > N \rightarrow |f(x) - L| < \varepsilon$$

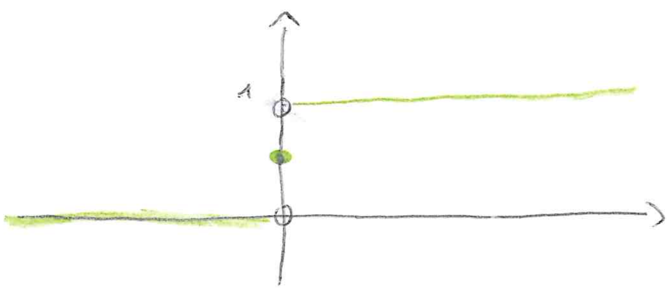
similarly:

$$\lim_{x \mapsto -\infty} f(x) = 1$$

Example 5:

$$\vartheta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

'Heaviside' step function



$\lim_{x \mapsto 0} \vartheta(x)$ is not defined.

How, if I come from the 'left':

$$\lim_{x \mapsto 0^-} \vartheta(x) = 0$$

and, if I come from the right

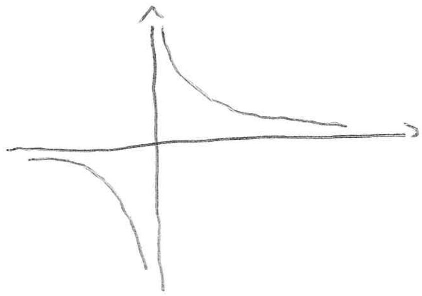
$$\lim_{x \mapsto 0^+} \vartheta(x) = 1$$

$\lim_{x \mapsto x_0^-} f(x) = L \qquad \forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - x_0| < \delta \text{ and } x < x_0 \rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \mapsto x_0^+} f(x) = L \qquad \forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - x_0| < \delta \text{ and } x > x_0 \rightarrow |f(x) - L| < \varepsilon$

Example 6

$f(x) = \frac{1}{x}$

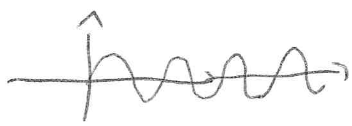


$\lim_{x \mapsto 0^+} f(x) = +\infty$ $\quad$ ($\lim_{x \mapsto 0^+} f(x) = \infty$: $\forall N > 0\ \exists\ \delta > 0 / 0 < |x - x_0| < \delta$ and $x > x_0 \rightarrow f(x) > N$

$\lim_{x \mapsto 0^-} f(x) = -\infty$

Example 7:

The limit does not always exist..

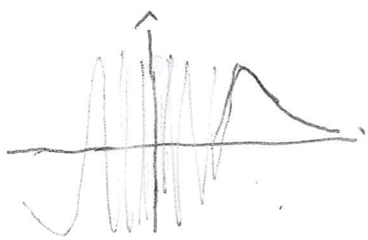


$\lim_{x \mapsto \infty} \sin x$?

We only know that $|\sin x| < 1$, but there is not a value to which it tends

Similarly;

$\lim_{x \mapsto 0} \operatorname{sen}\left(\frac{1}{x}\right)$



Ndet.

"fast up and down"

General properties of limits:

$$\lim_{x \mapsto x_0} f(x) = L_1 \; ; \; \lim_{x \mapsto x_0} g(x) = L_2$$

then: $\forall \alpha, \beta \in R$

1) $$\lim_{x \mapsto x_0} \left(\alpha f(x) + \beta g(x) \right) = \alpha L_1 + \beta L_2$$

2) $$\lim_{x \mapsto x_0} f(x) \cdot g(x) = L_1 \cdot L_2$$

3) $$\lim_{x \mapsto x_0} \frac{1}{f(x)} = \frac{1}{L_1}$$

4) $$\lim_{x \mapsto x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$

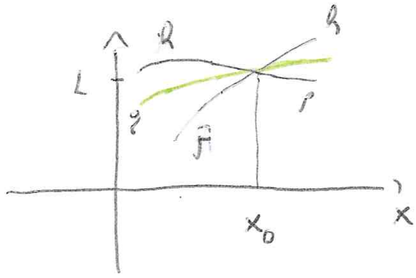
Squeeze theorem (Einschnürungssatz) (Teorema dei carabinieri) ☺

$$f(x) \leq g(x) \leq h(x) \qquad \forall x \in D, \; x \neq x_0$$

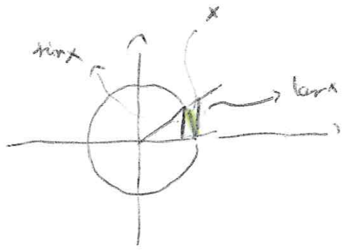
$$\lim_{x \mapsto x_0} f(x) = \lim_{x \mapsto x_0} h(x) = L$$

$$\Downarrow$$

$$\lim_{x \mapsto x_0} g(x) = L$$



Important limits :



$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\sin x < x < \tan x \longrightarrow 1 < \frac{x}{\sin x} < \frac{\tan x}{x} = \frac{1}{\cos x}$$

$$\lim_{x \to 0} 1 = 1 ; \quad \lim_{x \to 0} \frac{1}{\cos x} = 1 \Rightarrow \lim_{x \to 0} \frac{x}{\sin x} = 1 \Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{1} = 1 ; \text{ q.e.d}$$

$$\lim_{x \to 0} \frac{\sin(\alpha x)}{\beta x} = \frac{\alpha}{\beta}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} \frac{\arctan x}{x} = 1$$

$$\lim_{x \to 0} \frac{\arcsin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

$$\lim \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a, \; a > 0$$

$f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$

$x_0 \in D$;

def: $f(x)$ is continuous in $x = x_0$ if $\lim_{x \mapsto x_0} f(x) = f(x_0)$.

Examples:

- $f(x) = x^2$ is continuous everywhere ... (for instance, for $x_0 = 2$: $\lim_{x \mapsto 2} f(x) = 4 = f(x_0)$

- $f(x) = \begin{cases} x^2 & x \neq 2 \\ 5 & x = 2 \end{cases}$ is not continuous for $x = 2$: $\lim_{x \mapsto 2} f(x) = 4 \neq f(2) = 5$

- $\vartheta(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$ is not cont. for $x = 0$.