$\psi(q,x) = N\sqrt{1-x^2}$ for $-1 \leq x \leq 1$.

a) $$\int_{-\infty}^{\infty} |\varphi|^2 dx = |N|^2 \int_{-1}^{1} dx\,(1-x^2) = |N|^2 \left[x - \frac{x^3}{3}\right]_{-1}^{1} = |N|^2 \left(2\left(1-\frac{1}{3}\right)\right)$$

$$= |N|^2 \cdot \frac{4}{3} = 1$$

$|N| = \sqrt{\frac{3}{4}}$ $\quad N = \frac{\sqrt{3}}{2} e^{i\phi}$ whereas ϕ is a real number.

b)

$$\text{Probability} = \int_{-1/2}^{1/2} |\psi|^2 dx = |N|^2 \left[x - \frac{x^3}{3}\right]_{-1/2}^{1/2} = |N|^2 \left[2\left(\frac{1}{2} - \frac{1}{8\cdot 3}\right)\right] =$$

$$= |N|^2 \cdot 2\,\frac{12-1}{24} = |N|^2 \frac{11}{12} = \frac{\overset{1}{\cancel{3}}}{4}\,\frac{11}{\cancel{12}_4} = \frac{11}{16}$$

prob to find the particle between $-\frac{1}{2}$ and $\frac{1}{2}$ $= \frac{11}{16}$

c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\psi|^2 dx = 0$$ because $|\psi|^2$ is even, $x|\psi|^2$ is odd!

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = |N|^2\int_{-1}^{1} x^2(1-x^2)\,dx = |N|^2 \cdot \left(\frac{x^3}{3} - \frac{x^5}{5}\right)_{-1}^{1} =$$

$$= |N|^2\left(2\left(\frac{1}{3}-\frac{1}{5}\right)\right) = \frac{3}{4}\cdot 2 \cdot \frac{5-3}{15} = \frac{\cancel{3}^{1}}{\cancel{4}}\cdot 8 \cdot \frac{2}{\cancel{15}5} = \frac{1}{5}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{\sqrt{5}}$$

d) $p = p_x = -i\hbar\partial_x$

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi(p\psi)\,dx$$

$$p\psi = -i\hbar\frac{\partial}{\partial x}\sqrt{1-x^2}\cdot N = -iN\hbar\cdot\frac{-2x}{\sqrt{1-x^2}} = 2iN\hbar\frac{x}{\sqrt{1-x^2}}$$

$\langle p\rangle = 0!$

$$P_x^2 = P^2 = \int_{-1}^{1} \psi P^2 \psi$$

$$P^2\psi = (-i\hbar)^2 \partial_x^2 \psi = -\hbar^2 \cdot N \cdot \partial_x \frac{-2x}{\sqrt{1-x^2}} =$$

$$= \hbar^2 N \partial_x \left(\frac{2x}{\sqrt{1-x^2}}\right) = \hbar^2 N \frac{2\sqrt{1-x^2} - 2x \frac{-2x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$= \hbar^2 N \frac{2(1-x^2) + 4x^2}{(1-x^2)^{3/2}} = \hbar^2 N \frac{2(x^2+1)}{(1-x^2)^{3/2}}$$

Ergo:

$$\langle P^2 \rangle = |N|^2 \hbar^2 \int_{-1}^{1} \sqrt{1-x^2} \cdot \frac{2(x^2+1)}{\sqrt{1-x^2}\,(1-x^2)} = |N|^2 \hbar^2 \int_{-1}^{1} \frac{2(x^2+1)}{(1-x^2)} dx$$

NO NEED TO SOLVE! THE INTEGRAL DIVERGES! $x \mapsto 1^-$ and $x \mapsto -1^+$, you get a divergence! Ergo:

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \sqrt{\langle P^2 \rangle} = |N| \hbar \sqrt{\int_{-1}^{1} \frac{2(x^2+1)}{(1-x^2)} dx}$$

$$= \infty!$$

$\Delta x \Delta P = \infty$ → Uncertainty relation fulfilled :)

a) $|\alpha|^2 + |\beta|^2 + 3|\gamma|^2 = 1$

b) $\alpha = e^{i\varphi}$, $\beta = \gamma = 0$ $\Rightarrow$ Eigenvalue: $E_1 = -\frac{E_0}{n^2} = -E_0$

or

$\alpha = 0$, $\beta, \gamma \neq 0$ with $|\beta|^2 + 3|\gamma|^2 = 1$. $\Rightarrow$ Eigenvalue: $E_2 = -\frac{E_0}{4}$.

c) α, β and $\gamma = 0$ $\Rightarrow$ $\vec{L}^2 = 0$ (eigenvalue)
with
$|\alpha|^2 + |\beta|^2 = 1$

$\alpha = \beta = 0$, $\gamma = e^{i\varphi}$/Sqrt[3] $\Rightarrow$ $\vec{L}^2 = \hbar^2 1\cdot(1+1) = 2\hbar^2$ (eigenvalue

d) $\alpha, \beta \neq 0$, $\gamma = 0$
$|\alpha|^2 + |\beta|^2 = 1$ $\Rightarrow$ $L_z = 0$

e) $\gamma = \pm 1/\sqrt{6}$
$\beta = \pm 1/\sqrt{2}$
$\alpha = 0$

1) $H[\vec{n}] = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + V(\vec{x})$, $V(\vec{x}) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)$

2) We need to evaluate for 1 particle the integral:

$$\int d^3x\, e^{-\beta V(\vec{x})} = \int dx\,dy\,dz\, e^{-\frac{\beta}{2} m\omega^2 (x^2+y^2+z^2)}$$

$$= \left[\int dx\, e^{-\frac{\beta}{2} m\omega^2 x^2}\right]^3 =$$

$$= \left[\sqrt{\frac{2\pi}{\beta m\omega^2}}\right]^3$$

This means that

$$Z[T,\omega,N] = \frac{1}{\lambda^{3N} N!} \left[\sqrt{\frac{2\pi}{\beta m\omega^2}}\right]^{3N}$$

whereas:

$$\lambda = \sqrt{\frac{2\pi\hbar^2\beta}{m}}$$

Therefore, one obtains:

$$Z[T,\omega,N] = \frac{1}{N!}\left[\frac{1}{\sqrt{\frac{2\pi\hbar^2\beta}{m}}}\sqrt{\frac{2\pi}{\beta m\omega^2}}\right]^{3N}$$

$$= \frac{1}{N!}\left[\frac{1}{\hbar}\,\frac{1}{\beta\omega}\right]^{3N}$$

3) Free energy:

$$F(T,\omega,N) = -k_B T \ln Z[T,\omega,N]$$

$$= -k_B T\left(3N\ln\left[\frac{1}{\hbar\beta\omega}\right] - \ln N!\right)$$

$$\boxed{F(T,\omega,N) = -k_B T\left(3N\ln\left(\frac{k_B T}{\hbar\omega}\right) - \ln N!\right)}$$

4) $$S = -\frac{\partial F}{\partial T} = +k_B\left(3N\ln\left(\frac{k_B T}{\hbar\omega}\right) - \ln N!\right) + k_B T\left(\frac{3N}{T}\right) = 3N k_B \ln\left(\frac{k_B T}{\hbar\omega}\right) + \text{const}$$

4.21

4.2.1)

$$\mathcal{Z}[T, \omega, \mu] = \sum_{N=0}^{\infty} e^{\beta \mu N} \mathcal{Z}[T, \omega, N] .$$

Being:

$$\mathcal{Z}[T, \omega, N] = \frac{1}{N!} \left[\frac{K_B T}{\hbar \omega} \right]^{3N} ,$$

one has:

$$\mathcal{Z}_{GK} = \sum_{N=0}^{\infty} e^{\beta \mu N} \frac{1}{N!} \left[\frac{K_B T}{\hbar \omega} \right]^{3N} =$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \left[\left(\frac{K_B T}{\hbar \omega} \right)^3 e^{\beta \mu} \right]^N = e^{\left(\frac{K_B T}{\hbar \omega} \right)^3 e^{\beta \mu}}$$

$$\boxed{\mathcal{Z}_{GK} = \mathcal{Z}[T, \omega, \mu] = e^{\left(\frac{K_B T}{\hbar \omega} \right)^3 e^{\frac{\mu}{K_B T}}}}$$

4.2.2)

$$\ln Z_{GK} = \left(\frac{k_B T}{\hbar\omega}\right)^3 e^{\frac{\mu}{k_B T}}$$

$$\langle N \rangle = k_B T \frac{\partial}{\partial\mu} \ln Z_{GK} = \frac{(k_B T)^4}{(\hbar\omega)^3} \frac{1}{(k_B T)} e^{\frac{\mu}{k_B T}} = \left(\frac{k_B T}{\hbar\omega}\right)^3 e^{\frac{\mu}{k_B T}}$$

$$\boxed{\langle N \rangle = \left(\frac{k_B T}{\hbar\omega}\right)^3 e^{\frac{\mu}{k_B T}}}$$

4.2.3

$$\mathcal{E} = -\frac{\partial}{\partial\beta} \ln Z_{GK} = k_B T^2 \frac{\partial}{\partial T} \ln Z_{GK} =$$

$$= k_B T^2 \left[\frac{k_B}{\hbar\omega} \cdot 3 \left(\frac{k_B T}{\hbar\omega}\right)^2 e^{\frac{\mu}{k_B T}}\right]$$

$$= 3 \langle N \rangle k_B T$$