Rechenaufgabe 1.2

a) $$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$\int_{-\infty}^{\infty} |N|^2 e^{-\alpha^2 x^2} = |N|^2 \sqrt{\frac{\pi}{\alpha^2}} = 1 \longrightarrow \boxed{N = \sqrt{\frac{\alpha}{\sqrt{\pi}}} \; e^{i\phi}}$$

b) $$\langle x \rangle = \int_{-\infty}^{\infty} \underbrace{x|\psi|^2}_{\text{odd function}} dx \boxed{= 0 = \langle x \rangle}$$

$$\langle x^2 \rangle = |N|^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} = |N|^2 \left[-\frac{\partial}{\partial \lambda} \int_{-\infty}^{\infty} e^{-\lambda x^2} dx \right]_{\lambda = \alpha^2}$$

$$= |N|^2 \left[-\frac{\partial}{\partial \lambda} \sqrt{\frac{\pi}{\lambda}} \right]_{\lambda = \alpha^2} = |N|^2 \sqrt{\pi} \cdot \frac{1}{2} \left(\lambda^{-3/2} \right)_{\lambda = \alpha^2}$$

$$= |N|^2 \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\alpha^3} = \frac{\alpha}{\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{2} \frac{1}{\alpha^{3}} = \boxed{\frac{1}{2\alpha^2} = \langle x^2 \rangle}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle} = \boxed{\frac{1}{\sqrt{2}\,\alpha} = \Delta x}$$

c)

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty} \underbrace{\psi^* \partial_x \psi}_{\text{odd}}\, dx = 0$$

$$\langle p^2\rangle = \int \psi\, (-i\hbar\partial_x)^2\, \psi\, dx$$ could be calculated, but we do not do it.

Namely, notice that:

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle}$$ (Heis

and therefore because of

$$\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$$

$$\Delta p \geq \frac{\hbar}{2\cdot\frac{1}{\sqrt{2}\alpha}} = \frac{\hbar\alpha}{\sqrt{2}}$$

Ergo:

$$\sqrt{\langle p^2\rangle} \geq \frac{\hbar\alpha}{\sqrt{2}}$$

$$\boxed{\langle p^2\rangle \geq \frac{\hbar^2\alpha^2}{2}}$$

Rechenaufgabe 2.2

a) $\frac{1}{3} + |\beta|^2 = 1$

$|\beta|^2 = \frac{2}{3}$ $\qquad \beta = \sqrt{\frac{2}{3}}\, e^{i\phi}$

β real and positive $\rightarrow$ $\boxed{\beta = \sqrt{\frac{2}{3}}}$

b) $O_+ = |+\rangle\langle+|$

$|+\rangle$ is obviously eigenvector with eigenvalue 1

$|-\rangle$ " " " " " 0

c) One gets "1" with probability $\left|\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{3}$

" " "0" " " " $\left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$.

Just after the measurement the state is:

$$|\psi\rangle_{t=0^+} = \begin{cases} |+\rangle & \text{with prob. } \frac{1}{3} \\ |0\rangle & \text{" " } \frac{2}{3} \end{cases}$$

$\hookrightarrow$ The outcomes are the eigenvalues of the observable O_+: that is, "1" or "0".

$$|\psi_\perp\rangle = \sqrt{\tfrac{2}{3}}\,|+\rangle - \sqrt{\tfrac{1}{3}}\,|-\rangle$$

is $\perp$ to $|\psi\rangle$. In fact:

$$\langle\psi_\perp|\psi\rangle = 0 .$$

Rechenaufgabe 3.2:

a) $Z = \sum_n e^{-E_n/k_B T} = e^{-\frac{E_1}{k_B T}} + e^{-\frac{E_2}{k_B T}} + e^{-\frac{E_3}{k_B T}}$

b) $\hat{\rho} = \sum_n p_n |E_n\rangle\langle E_n|$ with $p_n = \frac{e^{-\beta E_n}}{Z}$

Ergo:

$\langle E_m | \hat{\rho} | E_k \rangle = p_m \delta_{mk}$ with $p_m = \frac{e^{-\beta E_m}}{e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}}$

Matrix form für $\hat{\rho}$:

$$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$$

c) Free Energy:

$$F = -k_B T \ln Z = -k_B T \ln \left(e^{-\frac{E_1}{kT}} + e^{-\frac{E_2}{kT}} + e^{-\frac{E_3}{kT}} \right)$$

$$\mathcal{E} = -\partial_\beta \ln Z \quad \text{with} \quad \ln Z = \ln \left(e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3} \right)$$

Ergo:

$$\boxed{\mathcal{E} = \frac{E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2} + E_3 e^{-\beta E_3}}{e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}}}$$

Entropy:

$$S = -\frac{F}{T} + \frac{\mathcal{E}}{T} = k_B \ln \left(e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3} \right) + \frac{1}{T} \left(\frac{E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2} + E_3 e^{-\beta E_3}}{e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}} \right).$$

d) N disting. particles:

$$Z = \left[e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}\right]^N$$

$$\ln Z = N \ln\left(e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}\right)$$

$$\boxed{F = -K_B T \ln Z = N F_1}$$

whereas $F_1 = -K_B T \ln\left(e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}\right)$

$$\ln Z_{tot} = -2V \int \frac{d^3p}{(2\pi)^3 \hbar^3} \ln\left(1 - e^{-\beta c |\vec{p}|}\right)$$

a)

$$\ln Z_{tot} = -2V \frac{4\pi}{8\pi^3 \hbar^3} \int_0^\infty p^2 dp \ln\left(1 - e^{-\beta c p}\right)$$

$$a = \beta c |\vec{p}|$$

$$da = \beta c dp$$

Ergo:

$$\ln Z_{tot} = \frac{V(-1)}{\pi^2 (\hbar c)^3 \beta^3} \int_0^\infty da\, a^2 \ln\left(1 - e^{-a}\right)$$

$$\boxed{\ln Z_{tot} = \frac{V (k_B T)^3}{(\hbar c)^3} \frac{\pi^2}{45}}$$

b) $$p = k_B T \frac{\partial \ln Z}{\partial V} = \frac{(k_B T)^4}{(\hbar c)^3} \frac{\pi^2}{45}$$

$$\mathcal{E} = k_B T^2 \frac{\partial}{\partial T} \ln Z = 3 \frac{(k_B T)^4}{(\hbar c)^3} \frac{\pi^2}{45}$$

$$s = \frac{\varepsilon + P}{T} = \frac{S}{V} = \frac{k_B (k_B T)^3}{(\hbar c)^3} \, 4 \, \frac{\pi^2}{45}$$

$$S = k_B \ln Z + k_B T \frac{\partial \ln Z}{\partial T}$$

$$= k_B \frac{V (k_B T)^3}{(\hbar c)^3} \frac{\pi^2}{45} + k_B T \cdot k_B \cdot \frac{3 (k_B T)^2}{(\hbar c)^3} \frac{\pi^2}{45}$$

$$= k_B \frac{V (k_B T)^3}{(\hbar c)^3} \frac{\pi^2}{45} \cdot 4 = s V$$

c)

$$\boxed{\varepsilon = T \frac{dP}{dT} - P}$$

$P = \alpha T^4$

$\varepsilon = 3 \alpha T^4 \rightarrow 3 \alpha T^4 = T \cdot 4 \alpha T^3 - \alpha T^4 = 3 \alpha T^4$

OK

it is fulfilled.

d) GAS OF PHOTONS! (BLACK-BODY)

IN FACT: MASSLESS PARTICLES WITH DEGENERACY EQUAL "2".