Rechenaufgabe 1

a) $w = \frac{1}{2}\varepsilon_0 \vec{E}^2 = \frac{1}{2}\varepsilon_0\, \theta(x)\theta(L_1-x)\theta(y)\theta(L_2-y)\theta(z)\theta(L_3-z)(A^2+B^2+C^2)$

(This is because $\theta^2(x) = \theta(x)$).

b) Die gesamte Energie lautet

$$E_{Feld} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, w =$$

$$= \int_0^{L_1} dx \int_0^{L_2} dy \int_0^{L_3} dz\, \frac{1}{2}\varepsilon_0 (A^2+B^2+C^2) = \frac{1}{2}\varepsilon_0 (A^2+B^2+C^2) L_1 L_2 L_3 .$$

c) Da $\vec{B} = \vec{0}$ hat man $\vec{S} = \vec{0}$.

d) Bewegungsgleichung: $\vec{F} = q\vec{E}$

$$\begin{cases} m\ddot{x} = qA \\ m\ddot{y} = qB \\ m\ddot{z} = qC \end{cases} \mapsto \begin{cases} x = \frac{qA}{m} t^2 + \frac{L_1}{2} \\ y = \frac{qB}{m} t^2 + \frac{L_2}{2} \\ y = \frac{qC}{m} t^2 + \frac{L_3}{2} \end{cases} \quad \text{für } t \in (0, t_*).$$

Um t_* zu bestimmen, muss man aufpassen. Es muss nämlich gelten, dass:

$x \leq L_1 \wedge y \leq L_2 \wedge z \leq L_3$.

$x \leq L_1$ bedeutet: $\frac{qA}{m} t^2 + \frac{L_1}{2} \leq L_1 \mapsto t^2 \leq \frac{L_1}{2}\frac{m}{qA} \mapsto t \leq \sqrt{\frac{L_1}{2}\frac{m}{qA}}$.

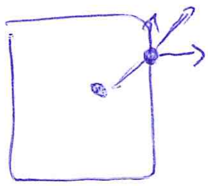
Ähnlich für y und z.

Letztendlich bedeutet es:

$$\begin{cases} x = \frac{qA}{m} t^2 + \frac{L_1}{2} \\ y = \frac{qB}{m} t^2 + \frac{L_2}{2} \\ z = \frac{qC}{m} t^2 + \frac{L_3}{2} \end{cases} \qquad \text{für } 0 \leq t \leq t_* = \min\left\{ \sqrt{\frac{L_1}{2}\frac{m}{qA}}, \sqrt{\frac{L_2}{2}\frac{m}{qB}}, \sqrt{\frac{L_3}{2}\frac{m}{qC}} \right\}$$

Für $t > t_*$ geht die Bewegung mit konstanter Geschwindigkeit fort:

$$\begin{cases} x = \frac{2qA}{m} t_* \cdot t + \cancel{\left(\frac{qA}{m} t_*^2 + \frac{L_1}{2}\right)} \tilde{X}_1 \\ y = \frac{2qB}{m} t_* t + \cancel{\left(\frac{qB}{m} t_*^2 + \frac{L_2}{2}\right)} + \tilde{X}_2 \\ z = \frac{2qC}{m} t_* t + \cancel{\left(\frac{qC}{m} t\right)} + \tilde{X}_3 \end{cases}$$



Für $\tilde{X}_1$:

$$\frac{2qA}{m} t_*^2 + \tilde{X}_1 = \frac{qA}{m} t_*^2 + \frac{L_1}{2}$$

$$\tilde{X}_1 = -\frac{qA}{m} t_*^2 + \frac{L_1}{2} \qquad \left(\text{wenn } t_* = \sqrt{\frac{L_1}{2}\frac{m}{qA}}, \text{ dann } \tilde{X}_1 = 0\right)$$

und so für $\tilde{X}_2$ und $\tilde{X}_3$.

$$A^\mu = b\,\varepsilon^\mu \cos\left(\frac{\omega t - kz}{\hbar}\right)$$

a) $\Box A^\mu = 0$

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0.$$

$$\partial_\nu A^\nu = 0 \mapsto \Box A^\mu = 0.$$

(siehe b)

achtung:

$$\Box A^\mu = 0 \mapsto \frac{\omega^2}{c^2} = k^2 \mapsto \boxed{\omega = kc\,!} \quad \left(\text{da } \Box = \frac{1}{c^2}\partial_t^2 - \Delta\right).$$

b) $$\partial_\nu A^\nu = \partial_y A^2 = \partial_y\left(b\cos\left(\frac{\omega t - kz}{\hbar}\right)\right) = 0.$$

c) $$E^i = c\,F^{i0} = c(\partial^i A^0 - \partial^0 A^i) = c(-\partial_x A^0 - \partial_0 A^i).$$

$A^0 = 0$, ergo

$$E^i = -c\,\partial_0 A^i.$$

$A^1 = A^3 = 0$. Nur $A^2 \neq 0$. Ergo:

$$E^1 = E^3 = 0.$$

$$E^2 = -c\,\partial_t\left(b\cos\left(\frac{\omega t - kz}{\hbar}\right)\right) = +cb\cdot\frac{\omega}{\hbar}\sin\left(\frac{\omega t - kz}{\hbar}\right)$$

$$\vec{E} = \left(0, \frac{bc\omega}{\hbar}\sin\left(\frac{\omega t - kz}{\hbar}\right), 0\right).$$

d)

$$F^{ij} = -\varepsilon^{ijk} B^k$$

$$F^{ij} = \partial^i A^j - \partial^j A^i$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \partial^1 A^2 = 0$$, da A^2 nicht von x abhängt

$$F^{13} = \partial^1 A^3 - \partial^3 A^1 = 0$$

$$F^{23} = \partial^2 A^3 - \partial^3 A^2 = -\partial^3 A^2 = \partial_3 A^2 = \partial_z A^2$$

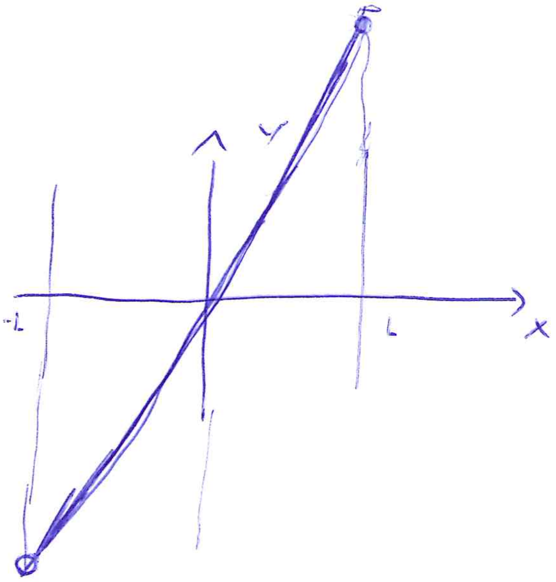
$$= \partial_z \left(b \cos\left(\frac{\omega t - kz}{\hbar}\right)\right) = \frac{bk}{\hbar} \sin\left(\frac{\omega t - kz}{\hbar}\right)$$

$$F^{23} = \cancel{\varepsilon^{123} B^1} = -\varepsilon^{231} B^1 = -B^1$$

$$B^1 = -\frac{bk}{\hbar} \sin\left(\frac{\omega t - kz}{\hbar}\right), \quad B^2 = B^3 = 0 .$$

$$\rho = \sigma_0\, \delta(2x-y)\, \Theta(4L^2-z^2)\, \Theta(L^2-x^2);$$

1) $\dfrac{xy\text{-Ebene}}{|x|<L}$

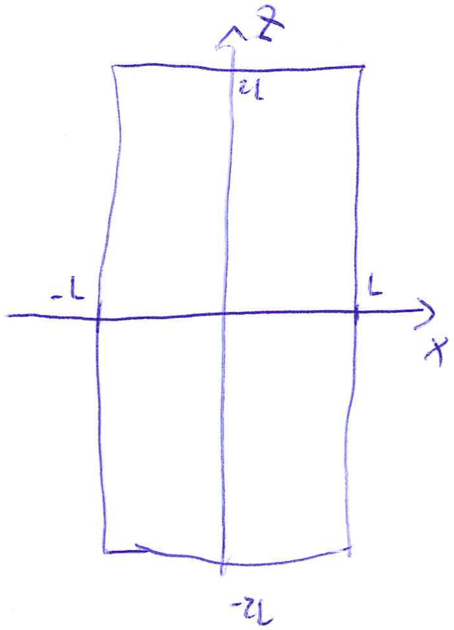


$2x = y$

2) xz- Ebene

$|z| < 2L$

$|x| < L$



c)
$$Q = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \sigma_0\, \delta(2x-y)\, \Theta(4L^2-z^2)\, \Theta(L^2-x^2)$$

$$= \sigma_0 \left(\int_{-2L}^{2L} dz \right) \cdot \left[\underbrace{\int_{-\infty}^{\infty} dx\, \Theta(L^2-x^2)}_{2L} \underbrace{\int_{-\infty}^{\infty} \delta(2x-y)\, dy}_{1} \right] = \sigma_0 \cdot 4L \cdot 2L$$
$$= 8\sigma_0 L^2.$$

check

$= \sigma_0 \cdot 4L \cdot 2L .$

—

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, \theta(L^2 - x^2) \frac{1}{2} \delta\left(\cancel{2}x - \frac{y}{2}\right)$$

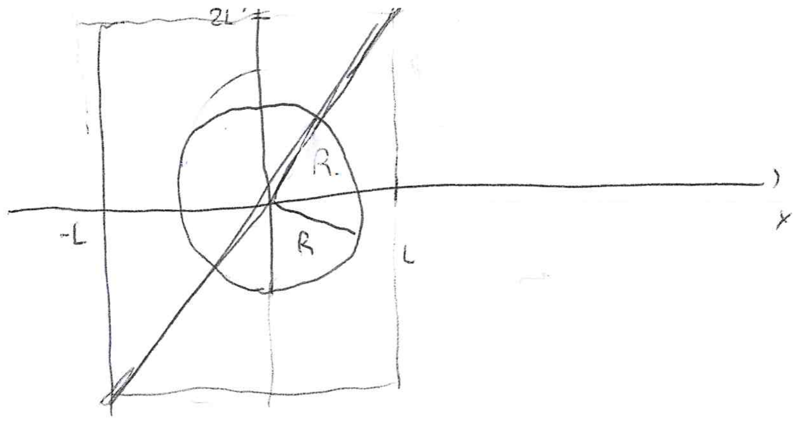
$|x| < L$

$$\int_{-\infty}^{\infty} dx \, \theta(L^2 - x^2) \frac{1}{2} \delta\left(x - \frac{y}{2}\right) = \begin{cases} \frac{1}{2} & \text{für } |y| < 2L \\ 0 & \text{für } |y| > 2L \end{cases}$$

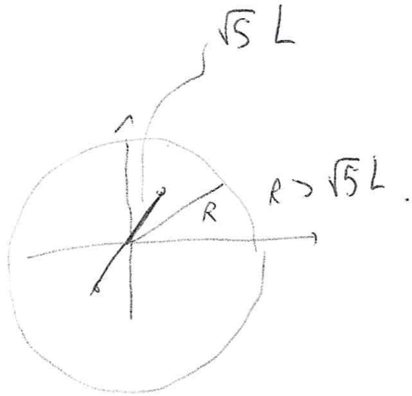
Ecco rimane

$$\frac{1}{2} \int_{-2L}^{2L} dy = 2L$$

ok.

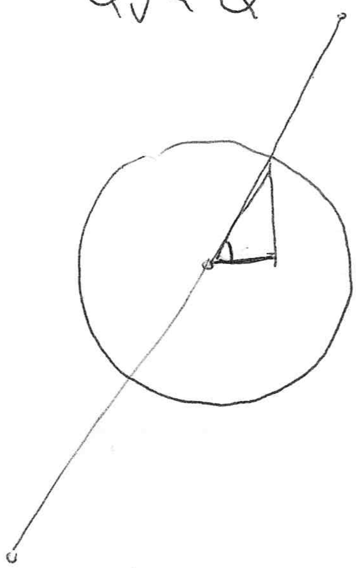


$Q_V =$?



if $R > \sqrt{5}\, L$ $\quad Q_V = Q \mapsto \varphi = \frac{Q}{\varepsilon_0}$

if $R < \sqrt{5}\, L$ $\quad Q_V < Q$



$x_{max} = R \cos\alpha$; Aber $\cos\alpha = \frac{1}{\sqrt{5}} \mapsto x_{MAX} = \frac{R}{\sqrt{5}}$

$$Q_V = \sigma_0 \int_{-2L}^{2L} dz \int_{-\infty}^{\infty} dx\, \theta\left(\frac{R^2}{5} - x^2\right) = \sigma_0 \cdot 4L \cdot \left(\frac{2R}{\sqrt{5}}\right) = 8\,\sigma_0 L \frac{R}{\sqrt{5}}$$

for $R < \sqrt{5}\, L$.

Check: for $R = \sqrt{5}\, L$ $\quad Q_V = Q$ as expected!