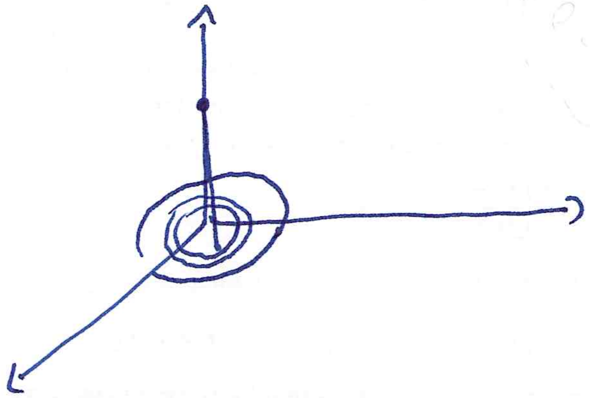
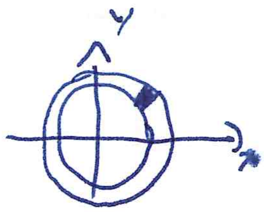
a) $\rho(\vec{r}) = \rho_0 \delta(z)\, \vartheta(R^2 - x^2 + y^2)$

$$\rho_0 = \frac{Q}{\pi R^2}.$$

b)





$$d\varphi = K \frac{dq}{\sqrt{r^2+z^2}} \qquad dq = \frac{Q}{\pi R^2} \cdot dr \cdot r\, d\varphi$$

$$d\varphi = \frac{K}{\sqrt{r^2+z^2}} \frac{Q}{\pi R^2}\, r\, dr\, d\varphi$$

$$\varphi(z) = \frac{KQ}{\pi R^2} \int_0^{2\pi} d\varphi \int_0^R \frac{r}{\sqrt{r^2+z^2}}\, dr$$

$$= \frac{KQ}{\pi R^2} \cdot 2\pi \cdot \int_0^R \frac{r}{\sqrt{r^2+z^2}}\, dr$$

The integral is:

$$\int_0^R \frac{r}{\sqrt{r^2+z^2}}\,dr$$

$$W = r^2 + z^2$$

$$dW = 2r\,dr$$

$$\frac{1}{2}\int_{z^2}^{R^2+z^2} dW\, W^{-1/2} = \left[\frac{1}{2}\,\frac{W^{\frac{1}{2}}}{\frac{1}{2}}\right]_{z^2}^{R^2+z^2} = \left[W^{1/2}\right]_{z^2}^{R^2+z^2} = \sqrt{R^2+z^2} - z$$

(für $z > 0$ sonst $|z|$)

Ergo:

$$\varphi(z) = \frac{2KQ}{R^2}\cdot\left(\sqrt{R^2+z^2} - z\right)$$

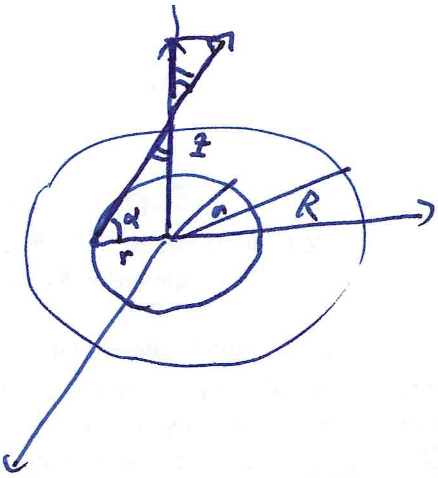
- $\vec{E} = ?$

$$E_z = -\frac{d\varphi}{dz} = -\frac{2KQ}{R^2}\left(\frac{\cancel{2} z}{\cancel{2}\sqrt{R^2+z^2}} - 1\right)$$

- die gesamte Ladung sei Q.
- $\vec{J} = \vec{0}$! (or can I have $\vec{B} = 0$ even if $\vec{J} \neq \vec{0}$?)
- Maybe 4 Punkte? and 2 Rechnen?

check:

(different way to calculate it)



$E_z = E \cdot \operatorname{sen}\alpha$

$$dE_z = K \frac{dq}{r^2+z^2} \frac{z}{\sqrt{r^2+z^2}} \qquad z = \operatorname{sen}\alpha \cdot \sqrt{z^2+r^2}$$

$$dq = r\,dr\,d\varphi \cdot \frac{Q}{\pi R^2}$$

$$dE_z = \frac{KQz}{\pi R^2} \frac{r\,dr}{(r^2+z^2)^{3/2}} d\varphi$$

$$E_z = 2\pi \frac{KQz}{\pi R^2} \cdot \int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

$w = r^2+z^2$

$dw = 2r\,dr$

$$\frac{1}{2}\int_{z^2}^{R^2+z^2} dw\; w^{-3/2} = \left(\frac{1}{2}\frac{w^{-1/2}}{-1/2}\right)_{z^2}^{R^2+z^2} = \left(-w^{-1/2}\right) = -\cancel{\tfrac{1}{2}}\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z}$$

ago we get.

$$E_z = 2\pi \frac{KQ}{\pi R^2} z \left(-\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z} \right)$$

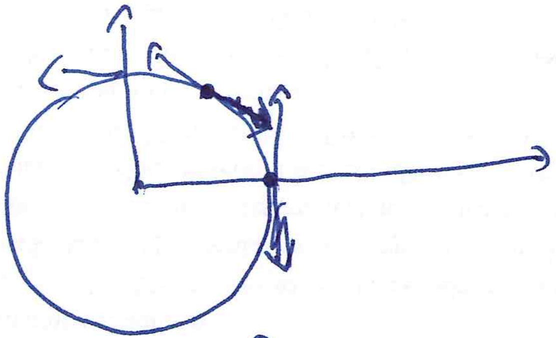
Yeah... it is really the same!

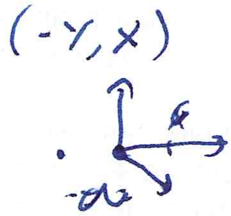
It works!

2)

a) $\vec{B} = b(-y, x, 0)\, e^{-\lambda(x^2+y^2)}$

$$\oint_K \vec{B}\cdot d\vec{s}$$





$$I = \oint_K \vec{B}\cdot d\vec{s} = B \cdot \int_0^{2\pi R} ds = B \cdot 2\pi R$$

(but s goes from 0 to $2\pi R$)

$$|\vec{B}| = b\, e^{-\lambda(x^2+y^2)} \cdot \sqrt{x^2+y^2}$$

ergo

$$|\vec{B}| = b\, e^{-\lambda R^2} \cdot R$$

$$\boxed{I = 2\pi\, b\, R^2 e^{-\lambda R^2}}$$

b) It must be

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

setting $A_x = A_y = 0$ we get:

$$\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & A_z \end{vmatrix},$$

out of which:

$$\begin{cases} B_x = \partial_y A_z \\ B_y = -\partial_x A_z \\ B_z = 0 \end{cases}.$$

(iii) Way 1: "guess":

$$\boxed{A_z = \frac{b}{2\lambda} e^{-\lambda(x^2+y^2)} + \text{const}}$$

check:

$$\begin{cases} B_x = \partial_y A_z = -by e^{-\lambda(x^2+y^2)} \\ B_y = -\partial_x A_z = bx e^{-\lambda(x^2+y^2)} \\ B_z = 0 \end{cases}$$

(ii) Way 2: "calculate"

Introduce $w = x^2 + y^2$ and $A_z = A_z(w)$

$$B_x = \frac{dA_z}{dw}\frac{\partial w}{\partial y} = 2y\frac{dA_z}{dw}.$$

$$B_y = -2x\frac{dA_z}{dw}$$

One has: $B_x = -bye^{-\lambda(x^2+y^2)}$, ergo:

$$2\frac{dA_z}{dw} = -be^{-\lambda w}$$

$$\Downarrow$$

$$\frac{dA_z}{dw} = -\frac{b}{2}e^{-\lambda w}$$

$$A_z = \frac{b}{2\lambda}e^{-\lambda w} + \text{const}$$

$$\boxed{A_z = \frac{b}{2\lambda}e^{-\lambda(x^2+y^2)} + \text{const}}$$

Calculation of $\vec{J}$. To that end we need

$$\boxed{-\Delta \vec{A} = \mu_0 \vec{J}}$$

$$\vec{J} = -\frac{1}{\mu_0} \Delta \vec{A}$$

$$J_x = J_y = 0 \;!$$ ok.

But

$$J_z = -\frac{1}{\mu_0}\left(\partial_x^2 A_z + \partial_y^2 A_z\right)$$

$$\partial_x A_z = -bx e^{-\lambda(x^2+y^2)} \qquad \partial_y A_z = -by e^{-\lambda(x^2+y^2)}$$

$$\partial_x^2 A_z = -b e^{-\lambda(x^2+y^2)} + 2b\lambda \cdot x^2 e^{-\lambda(x^2+y^2)}$$

$$\partial_y^2 A_z = -b e^{-\lambda(x^2+y^2)} + 2b\lambda y^2 e^{-\lambda(x^2+y^2)}$$

Ergo:

$$\boxed{J_z = -\frac{1}{\mu_0}\left(-2b e^{-\lambda(x^2+y^2)} + 2b\lambda (x^2+y^2) e^{-\lambda(x^2+y^2)}\right)}$$

ok.

a) $A^\mu = b\,\varepsilon^\mu e^{-\lambda(\omega t - kz)^2}$. $\varepsilon^\mu = (0,1,0,0)$

$$\partial_\mu A^\mu = \partial_t A^0 + \partial_x A^1 + \partial_y A^2 + \partial_z A^3 = \partial_x A^1 = 0.$$

in fact, $A^0 = A^2 = A^3 = 0$ and A^1 does not depend on x.

This is the Lorentz-gauge with $\partial_\mu A^\mu = 0$.

(In this case, being $A^0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$, also the Coulomb-gauge is realised).

b) The equation of motion is

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0.$$

Being $\partial_\nu A^\nu = 0$ we get:

$$\Box A^\mu = 0.$$

$\Box A^\mu = 0$ is authomatically fulfilled for $\mu = 0,2,3$ because

$$A^0 = A^2 = A^3 = 0.$$

The only non-trivial eq is

$$\Box A^1 = \frac{1}{c^2}\partial_t^2 A^1 - \Delta A^1 = \frac{1}{c^2}\partial_t^2 A^1 - \partial_z^2 A^1 = 0.$$

$$A^1 = b\, e^{-\lambda(\omega t - kz)^2}$$

$$\partial_t A^1 = b \cdot \left(-2\lambda(\omega t - kz)\cdot \omega\right) e^{-\lambda(\omega t - kz)^2}$$

$$= -2\lambda b\, \omega\, (\omega t - kz)\, e^{-\lambda(\omega t - kz)^2}.$$

$$\partial_t^2 A^1 = -2\lambda b \omega^2 e^{-\lambda(\omega t - kz)^2} + b\cdot(2\lambda\omega)^2 (\omega t - kz)^2 \cdot e^{-\lambda(\omega t - \ldots)^2}$$

$$\partial_z A^1 = b\left(-2\lambda(\omega t - kz)(-k)\right) e^{-\lambda(\ldots)^2}$$

$$= 2\lambda b k (\omega t - kz) e^{-\lambda(\ldots)^2}$$

$$\partial_z^2 A^1 = -2\lambda b k^2 e^{-\lambda(\ldots)^2} + b(2\lambda k)^2 (\omega t - kz)^2 e^{-\lambda(\ldots)^2}.$$

Ergo:

$$-2\lambda b \frac{\omega^2}{c^2} + b\frac{(2\lambda\omega)^2}{c^2}(\omega t - kz)^2 = -2\lambda b k^2 + b(2\lambda k)^2 (\omega t - kz)^2$$

Solution:

$$\boxed{\frac{\omega^2}{c^2} = k^2 \mapsto \omega = \pm c k.}$$

c) $F^{i0} = \frac{1}{c} E^{i0}$

In our case:

$$\frac{1}{c} E^1 = F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\partial^0 A^1 = -\frac{1}{c}\partial_t A^1 .$$

$$\boxed{E^1 = -\partial_t A^1 = 2\lambda b\omega (\omega t - kz) e^{-\lambda(\omega t - kz)^2}} .$$

other components:

$$\boxed{\begin{aligned} E^2 &= cF^{20} = 0 \\ E^3 &= cF^{30} = 0 \end{aligned}}$$

$$\vec{E} = (E^1 \neq 0, 0, 0) .$$

d) $F^{ij} = -\varepsilon^{ijk} B^k$

$$F^{12} = -B^3 = \partial^1 A^2 - \partial^2 A^1 = 0 \quad , \quad F^{23} = -B^1 = 0$$

$$F^{13} = -\varepsilon^{132} B^2 = B^2 .$$

$$B^2 = \partial^1 A^3 - \partial^3 A^1 = -\partial^3 A^1 = \partial_z A^1$$

$$\boxed{B^2 = 2\lambda b k (\omega t - kz) e^{-\lambda(\omega t - kz)^2}}$$

$$\boxed{B^1 = B^3 = 0}$$