The Finite Temperature Effective Potential for Local Composite Operators

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Abstract

The method of the effective action for the composite operators $\Phi^2(x)$ and $\Phi^4(x)$ is applied to the termodynamics of the scalar quantum field with $\lambda\Phi^4$ interaction. An expansion of the finite temperature effective potential in powers of $\hbar$ provides successive approximations to the free energy with an effective mass and an effective coupling determined by the gap equations. The numerical results are studied in the space-time of one dimension, when the theory is equivalent to the quantum mechanics of an anharmonic oscillator. The approximations to the free energy show quick convergence to the exact result.

1 Introduction

Thermal properties of quantum field systems are important for a study of phase transitions in the big bang cosmology and for an interpretation of heavy ion collisions. These systems are characterised by a large coupling constant, thus non-perturbative methods should be developped to study their termodynamics. Here we shall discuss an application of the effective action for local composite operators [1] to the finite temperaturequantum field theory. We consider a scalar field with a classical action in n-dimensional Euclidean space-time given by

$$S[\Phi] = \int [\frac{1}{2}\Phi(x)(-\partial^2 + m^2)\Phi(x) + \lambda\Phi^4(x)]\, d^n x, \tag{1}$$

but the method can be easily extended to fermionic systems.

Recently, the diagrammatic rules for the effective action for the operator $\Phi^2(x)$ have been found [2] and the method has been extended to include the operator $\Phi^4(x)$ [3]. The lowest order approximation coincides with the Gaussian Effective Action resulting from the variational principle with Gaussian trial states [4]. Within the thermal field theory formalism this would correspond to the finite temperature Hartree approximation for the free energy [5]. An expansion of the effective action for local composite operators in powers of $\hbar$ provides thus a non-perturbative method for a systematic improvement of the Hartree approximation both at $T = 0$ and $T \neq 0$. An advantage of using local operators lies in the fact that calculation is simpler than for bilocal composite operators $\Phi(x)\Phi(y)$ [6] and other systematic methods which contain the Gaussian Effective Action as the lowest approximation (f.e. the optimized expansion [7]).

2 Finite temperature effective potential for composite operators

The vacuum functional for the composite operators $\Phi^2(x)$ and $\Phi^4(x)$ may be represented by a path integral

$$Z[J, K] = e^{\frac{1}{\hbar}W[J,K]} = \int D\Phi\, e^{\frac{1}{\hbar}\left[-S[\Phi] + \frac{1}{2}\int J(x)\Phi^2(x)\, d^n x + \frac{1}{24}\int K(x)\Phi^4(x)\, d^n x\right]} \tag{2}$$

and the effective action is defined through a Legendre transform

$$\Gamma[\Delta, \Lambda] = W[J, K] - \frac{\hbar}{2} \int J(x)\Delta(x)\, d^n x - \frac{\hbar^3}{24} \int K(x)\Lambda(x)\, d^n x$$
$$- \frac{\hbar^2}{8} \int K(x)\Delta^2(x)\, d^n x, \tag{3}$$

where the background fields are given by

$$\hbar\Delta(x) = 2\frac{\delta W}{\delta J(x)} = < \Phi^2(x) >_{J,K}$$
$$\hbar^3\Lambda(x) = 24\frac{\delta W}{\delta K(x)} - 3\hbar^2\Delta^2(x) = < \Phi^4(x) >_{J,K} - 3 < \Phi^2(x) >^2_{J,K} \tag{4}$$

and $< ... >_{J,K}$ denotes the expectation value in the presence of external sources J and K. The physical quantities have to be calculated at $J = K = 0$, or equivalently at the values of the background fields for which the gap equations

$$\frac{\delta\Gamma}{\delta\Delta(x)} = -\frac{\hbar}{2}J(x) - \frac{\hbar^2}{4}\Delta(x)K(x) = 0,$$
$$\frac{\delta\Gamma}{\delta\Lambda(x)} = -\frac{\hbar^3}{24}K(x) = 0 \tag{5}$$

are satisfied.

The path-integral quantization in Euclidean space-time enables us to study the quantum field theory and its equilibrium thermodynamics by the same "imaginary time" formalism, if the appropriate boundary conditions are chosen [8, 9]. The vacuum functional (2) defines the generating functional for composite Green's functions in quantum field theory, if the path integral is taken over the functions which approaches some constant as $x_0 \to \pm\infty$. If the set of functions is periodic with a period $\beta = \frac{1}{T}$ in the "imaginary time coordinate" x_0, the integral (2) has a meaning of the partition function at the temperature T, generalised to include interactions with the external currents J and K. The finite temperature effective potential for the composite operators $\Phi^2(x)$ and $\Phi^4(x)$ is defined as

$$V_\beta(\Delta, \Lambda) = - \left.\frac{\Gamma_\beta[\Delta, \Lambda]}{\beta \int d^{n-1}x}\right|_{\Delta=const, \Lambda=const}. \tag{6}$$

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The free energy density can be obtained as $F_\beta = V_\beta(\Delta_\beta, \Lambda_\beta)$, where the temperature dependent expectation values of the composite fields, Δ_β and Λ_β, which correspond to $J = K = 0$, are determined by a stationary point of $V_\beta(\Delta, \Lambda)$, according to (5).

Representing the connected generating functional by the series

$$W[J, K] = \sum_{k=0}^{\infty} \hbar^k W_{(k)}[J, K], \tag{7}$$

the effective action for composite operators (3) may be obtained by eliminating the sources

$$J = \sum_{k=0}^{\infty} \hbar^k J_{(k)}[\Delta, \Lambda], \quad \text{and} \quad K = \sum_{k=0}^{\infty} \hbar^k K_{(k)}[\Delta, \Lambda] \tag{8}$$

in favour of the expectation values of the composite fields

$$\Delta = \sum_{k=0}^{\infty} \hbar^k \Delta_{(k)}[J, K], \quad \text{and} \quad \Lambda = \sum_{k=0}^{\infty} \hbar^k \Lambda_{(k)}[J, K], \tag{9}$$

order by order in $\hbar$. The coefficients J_0 and K_0 are determined implicitly by the lowest-order relations

$$\Delta(x) = \Delta_0(x) = G(x, x) = \; (x) \; \oint \bigcirc \tag{10}$$

and

$$\Lambda(x) = \Lambda_0(x) = \int G^4(x, y)(K_0(y) - 24\lambda) d^n y, \tag{11}$$

where the inverse propagator is given by

$$G^{-1}(x, y) = (-\partial^2 + \Omega^2(x))\delta(x - y), \tag{12}$$

with an effective mass $\Omega(x)$ defined by

$$\Omega^2[\Delta] = m^2 - J_0[\Delta]. \tag{13}$$

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Higher coefficients, $J_{(k)}$ and $K_{(k)}$ may be easily expressed as functionals of $J_0[\Delta]$ and $K_0[\Delta, \Lambda]$ and the Legendre transform (3) can be performed. To the order $\hbar^4$ the result is [3]

$$\Gamma[\Delta, \Lambda] = \int [\tfrac{\hbar}{2}(\Omega^2(x) - m^2)\Delta(x) - 3\lambda\hbar^2\Delta^2(x) - \hbar^3\lambda\Lambda(x)]dx$$

$$-\frac{\hbar}{2}\ \bigcirc\ -\frac{\hbar^3}{48}\ \diagram\ +\frac{\hbar^4}{48}\ \diagram\ + \ ... \quad (14)$$

with Ω related to Δ by (10) and (12), and the effective coupling

$$\diagram\ =\ \diagram\ \Lambda, \quad (15)$$

where a slash denotes an inversion of the operator.

The effective action which includes the two-particle operator $\Phi^2(x)$ only, $\Gamma[\Delta]$, can be obtained from the above expression for $\Gamma[\Delta, \Lambda]$ by setting $K_0 = 0$ which results in

$$\Gamma[\Delta] = \int [\tfrac{\hbar}{2}(\Omega^2(x) - m^2)\Delta(x) - 3\lambda\hbar^2\Delta^2(x)]dx$$

$$-\frac{\hbar}{2}\ \bigcirc\ +\frac{\hbar^3}{48}\ \diagram\ +\frac{\hbar^4}{48}\ \diagram\ + \ ... \quad (16)$$

in agreement with diagrammatic rules proven in Ref. [2].

The expansions of $\Gamma[\Delta]$ and $\Gamma[\Delta, \Lambda]$ in powers of $\hbar$ provide two different approximation schemes for physical quantities, we shall denote these as I2 and I4, respectively. The non-perturbative character of the approximations is due to the fact that the expectation values of the composite fields are determined from the gap equations, obtained by requiring the given order expression for the effective action to be stationary. At order $\hbar^2$ both approaches results in the Gaussian effective action [4], which contains an infinite sum of bubble diagrams of perturbation theory. As it happens, the approximations

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to physical quantities obtained from $\Gamma[\Delta]$ and $\Gamma[\Delta, \Lambda]$ coincide also at order $\hbar^3$, after solving the gap equations, and differences first appear at order $\hbar^4$. One can note that by introducing higher composite operators new approximation schemes can be formulated, but their influence will appear in much higher orders of $\hbar$.

The finite temperature effective potential for composite operators can be obtained from the effective action by replacing the Feynman rules by those at finite temperature. In the given order approximation to the effective action the infinite interval in all integrals over t is replaced by $[0, \beta]$. Δ and Λ are taken to be space-time independent, in this case Ω is also constant and the Fourier transform of the propagator (12) can be performed. Using a momentum representation in space, but a coordinate representation in time we have

$$G_\beta(\tau, \mathbf{p}) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \frac{\exp[-i\omega_m(\tau)]}{\mathbf{p}^2 + \omega_m^2 + \Omega^2} \tag{17}$$

where $\omega_m = \frac{2\pi m}{\beta}$ are the Matsubara frequencies. The free energy calculated at the order $\hbar^2$, appears identical with that in the Hartree approximation, provided by the finite temperature Gaussian Effective Potential [5]. The post-Gaussian corrections can be determined by taking into account higher orders of the finite temperature effective potential for composite operators.

3   Free energy of the quantum-mechanical anharmonic oscillator

In the space-time of one dimension (time) the $\lambda\Phi^4$ theory is equivalent to a quantum mechanical anharmonic oscillator with a Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \lambda x^4. \tag{18}$$

After rescaling all quantities in terms of λ, only one dimensionless parameter $z = \frac{m^2}{2\lambda^{2/3}}$ remains; therefore, when discussing numerical results, we put $\lambda = 1$ without a loss of generality. The spectrum of the anharmonic oscillator can be calculated numerically and provides the simplest test for approximation methods in quantum field theory. The free energy, which contains information on the whole spectrum, can be used to study the reliability of field-theoretical methods at finite temperature. The conventional loop expansion

for the free energy coincides with termodynamic perturbation theory [10], the non-perturbative methods provide better approximations.

Approximations to the ground state energy and to the second and fourth excitation of the anharmonic oscillator, calculated with the use of operators $\Phi^2(x)$ and $\Phi^4(x)$ have been shown to converge quickly to the exact results [3]; we shall discuss therefore the approximations to the free energy calculated from the termal effective potential for these operators. The temperature dependent propagator (17) in one dimensional space-time becomes equal to

$$G_\beta(\tau) = \frac{\cosh[\frac{\Omega}{2}(|\tau| - \beta)]}{\Omega \sinh[\frac{\beta\Omega}{2}]}. \tag{19}$$

and the integrals over $\beta \geq t \geq 0$ in Feynman diagrams can be easily performed. We shall compare two approximation methods:
(I2) using the operator $\Phi^2(x)$:

$F_\beta = V_\beta(\Delta_\beta)$, where $V_\beta(\Delta)$ is calculated from Eq. (16) and Δ_β is a solution of the gap equation

$$\frac{\delta V_\beta}{\delta\Delta(x)} = 0 \tag{20}$$

(I4) using the operators $\Phi^2(x)$ and $\Phi^4(x)$:

$F_\beta = V_\beta(\Delta_\beta, \Lambda_\beta)$, where $V_\beta(\Delta, \Lambda)$ is calculated from 14 and Δ_β and Λ_β are obtained as a solution of the gap equations

$$\frac{\delta V_\beta}{\delta\Delta(x)} = 0 \quad \text{and} \quad \frac{\delta V_\beta}{\delta\Lambda(x)} = 0. \tag{21}$$

The non-perturbative character of the approximations to the free energy is kept by solving the gap equations non-perturbatively (if one expanded the solutions of the gap equations to the given order in $\hbar$, the results would coincide to that order in λ with that obtained in thermodynamic perturbation theory). In both cases we choose the solution of the gap equations with the largest positive value for Ω. The free energy for three values of the parameter $z = \frac{m^2}{2\lambda^{2/3}}$, $z = 5, 0, -1$, is shown as a function of the inverse temperature

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$\beta = \frac{1}{T}$ in Figs. (1)-(3), respectively. The results obtained from $V_\beta(\Delta)$ in tree lowest orders of $\hbar$ (I21, I22 and I23) are compared with that obtained from $V_\beta(\Delta, \Lambda)$ at order $\hbar^3$ (I43). The exact free energy was obtained as $F_\beta = -\frac{1}{\beta} \ln \sum_n e^{-\beta E_n}$ where the energy levels, E_n, have been calculated by a linear variational method, using harmonic oscillator wave functions with an appropriately chosen frequency [11].

The comparison shows that an expansion of the termal effective potential for composite operators in powers of $\hbar$ provides a method to improve the finite temperature Hartree approximation (I21) in a systematic way. Both the methods, I2 and I4, work very well for the single well anharmonic oscillator, the last being better in the whole range of temperatures. In both cases a quality of approximations becomes worse for decreasing values of z and in the double well case ($z < 0$) the methods are applicable only if the wells are not too deep ($z > -1$). For $z = -1$ (Figure 3) large discrepancies between the results of different orders of the method I2 and the exact value appear. The results obtained at order $\hbar^3$ with the use of the operators $\Phi^2(x)$ and $\Phi^4(x)$ (I43) are definitely better than that obtained with the operator $\Phi^2(x)$ only (I23). At low temperature discrepancies do not exceed 2%, but they grow with increasing temperature, where a contibution of higher excitations becomes more important. This can be understood from the analysis of the method I4 for the spectrum of the anharmonic oscillator [3], where it has been demonstated that the quality of the approximations becomes worse and worse for increasing excitation level. There is a hope that including higher composite operators will broaden the range of a good convergence further, making the method applicable even in the case of the double well potential with deeper wells. It will be also interesting to discuss an application of the local composite operators method to study thermodynamics of the scalar quantum field theory in the case of untrivial space dimension, we reserve this matter for a future publication.

Acknowledgement

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Figure captions

Figure 1. The free energy of the anharmonic oscillator at $z = \frac{m^2}{2\lambda^{\frac{2}{3}}} = 5$, plotted vs $\beta = \frac{1}{T}$. The successive orders approximations obtained from $V_\beta[\Delta]$ (*dashed line*, I21, I22, I23) and from $V_\beta[\Delta, \Lambda]$ (*dotted line*, I43) compared with the exact value (*solid line*).

Figure 2. As in Fig.1, but at $z = 0$.

Figure 3. As in Fig.1, but at $z = -1$.