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Solving Schwinger-Dyson equations by truncation in zero-dimensional scalar quantum field theory

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Three sets of Schwinger-Dyson equations, for all Green's functions, for connected Green's functions, and for proper vertices, are considered in scalar quantum field theory. A truncation scheme applied to the three sets gives three different approximation series for Green's functions. For the theory in zero-dimensional space-time the results for respective two-point Green's functions are compared with the exact value calculated numerically. The best convergence of the truncation scheme is obtained for the case of proper vertices.

Path-integral quantization is a convenient approach to derive Schwinger-Dyson (SD) equations of quantum field theory[1] (QFT). We consider a scalar field in n-dimensional Euclidean space-time with a classical action given by

$$S[\Phi]=\int d^nx\,[\tfrac{1}{2}\Phi(x)(-\partial^2+m^2)\Phi(x)+\lambda\Phi^4(x)]\ . \tag{1}$$

Full information on quantum theory is contained in generating functionals for Green's functions. We shall consider three cases: (i) the vacuum functional generating all Green's functions G_k, (ii) the generating functional for connected Green's functions W_k, and (iii) the effective action, which generates proper vertices Γ_k. These functionals can be used to obtain the SD equation for respective Green's functions.

(i) The vacuum functional is given by the path integral

$$Z[J]=\sum\frac{1}{k!}\int d^nx_1\cdots d^nx_k\,G_k(x_1,\ldots,x_k)J(x_1)\cdots J(x_k)=:N\int D\Phi\exp\left[-S[\Phi]+\int d^nx\,J(x)\Phi(x)\right]\ , \tag{2}$$

where the normalization factor N is chosen such that $Z[0]=1$. Using the identity

$$\int D\Phi\frac{\delta}{\delta\Phi}\exp\left[-S[\Phi]+\int d^nx\,J(x)\Phi(x)\right]=0\ , \tag{3}$$

we obtain a functional differential equation

$$\left[\int d^nv\,\Delta^{-1}(x,v)\frac{\delta}{\delta J(x)}+4\lambda\frac{\delta}{\delta J^3(x)}\right]Z[J]=J(x)Z[J]\ , \tag{4}$$

where

$$\Delta^{-1}(x,v)=(-\partial^2+m^2)\delta(x-v)\ .$$

Differentiating Eq. (4) with respect to J at $J=0$ results in an infinite hierarchy of SD equations for all Green's functions:

$$\begin{aligned}
&\int d^nv\,\Delta^{-1}(x,v)G_1(v)+4\lambda G_3(x,x,x)=0\ ,\\
&\int d^nv\,\Delta^{-1}(x,v)G_2(v,y)+4\lambda G_4(x,x,x,y)=\delta(x-y)\ ,\\
&\int d^nv\,\Delta^{-1}(x,v)G_3(v,y,w)+4\lambda G_5(x,x,x,y,w)=\delta(x-y)G_1(w)+\delta(x-w)G_1(y)\ ,\\
&\int d^nv\,\Delta^{-1}(x,v)G_4(v,y,w,z)+4\lambda G_6(x,x,x,y,w,z)=\delta(x-y)G_2(w,z)+\delta(x-w)G_2(y,z)+\delta(x-z)G_2(y,w)\ ,\\
&\cdots\ .
\end{aligned} \tag{5}$$

(ii) The generating functional for connected Green's functions W_k is defined as

$$W[J]=\sum\frac{1}{k!}\int d^nx_1\cdots d^nx_k\,W_k(x_1,\ldots,x_k)J(x_1)\cdots J(x_k)=:\ln Z[J]\ , \tag{6}$$

and satisfies the differential equation

$$\int d^nv\,\Delta^{-1}(x,v)\frac{\delta W}{\delta J(v)}+4\lambda\left[\left[\frac{\delta W}{\delta J(x)}\right]^3+3\frac{\delta^2W}{\delta J^2(x)}\frac{\delta W}{\delta J(x)}+\frac{\delta^3W}{\delta J^3(x)}\right]=J(x)\ , \tag{7}$$

which generates the connected SD equations:

$$\int d^n v\, \Delta^{-1}(x,v)W_1(v)+4\lambda[W_1^3(x)+3W_2(x,x)W_1(x)+W_3(x,x,x)]=0\ ,$$

$$\int d^n v\, \Delta^{-1}(x,v)W_2(v,y)+4\lambda[3W_1^2(x)W_2(x,y)+3W_2(x,x)W_2(x,y)+3W_3(x,x,y)W_1(x)+W_4(x,x,x,y)]=\delta(x-y)\ ,$$

$$\begin{aligned}\int d^n v\, \Delta^{-1}(x,v)W_3(v,y,w)&\\ +4\lambda[6W_1(x)W_2(x,w)W_2(x,y)&+3W_1^2(x)W_3(x,y,w)+3W_4(x,x,y,w)W_1(x)+3W_3(x,x,y)W_2(x,w)\\ &+3W_3(x,x,w)W_2(x,y)+3W_2(x,x)W_3(x,y,w)+W_5(x,x,x,y,w)]=0\ ,\end{aligned} \tag{8}$$

$$\begin{aligned}\int d^n v\, \Delta^{-1}(x,v)W_4(v,y,w,z)+4\lambda[6W_2(x,z)W_2(x,y)W_2(x,w)&+6W_1(x)W_3(x,y,z)W_2(x,w)\\ &+6W_1(x)W_2(x,y)W_3(x,w,z)+6W_1(x)W_2(x,z)W_3(x,y,w)+3W_1^2(x)W_4(x,y,w,z)\\ &+3W_5(x,x,y,w,z)W_1(x)+3W_4(x,x,y,w)W_2(x,z)+3W_4(x,x,y,z)W_2(x,w)\\ &+3W_4(x,x,w,z)W_2(x,y)+3W_3(x,x,y)W_3(x,w,z)+3W_3(x,x,w)W_3(x,y,z)\\ &+3W_3(x,x,z)W_3(x,y,w)+3W_2(x,x)W_4(x,y,w,z)+W_6(x,x,x,y,w,z)]=0\ ,\end{aligned}$$

$\cdots$.

(iii) The effective action is defined as a Legendre transform:

$$\Gamma[\phi]=\sum\frac{1}{k!}\int d^n x_1\cdots d^n x_k\Gamma_k(x_1,\ldots,x_k)\phi(x_1)\cdots\phi(x_k)=:W[J]-\int d^n x\, J(x)\phi(x)\ , \tag{9}$$

where $\phi(x)=\delta W/\delta J(x)$. The differential equation for the effective action

$$\begin{aligned}\int d^n v\, \Delta^{-1}(x,v)\phi(v)+4\lambda\left[\phi^3(x)+3\left[\frac{\delta^2\Gamma}{\delta\phi^2(x)}\right]^{-1}\phi(x)+\int d^n v\, d^n u\, d^n w\frac{\delta^3\Gamma}{\delta\phi(v)\delta\phi(u)\delta\phi(w)}\left[\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(v)}\right]^{-1}\right.&\\ \left.\times\left[\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(u)}\right]^{-1}\left[\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(w)}\right]^{-1}\right]=-\frac{\delta\Gamma}{\delta\Gamma(x)}&\end{aligned} \tag{10}$$

gives the SD hierarchy for proper vertices

$$\int d^n v\, \Delta^{-1}(x,v)\phi(v)+4\lambda\left[\phi^3(x)-3\Gamma_2^{-1}(x,x)\phi(x)-\int d^n v\, d^n u\, d^n w\, \Gamma_2^{-1}(x,v)\Gamma_2^{-1}(x,u)\Gamma_2^{-1}(x,w)\Gamma_3(v,u,w)\right]=0\ ,$$

$$\begin{aligned}\Delta^{-1}(x,y)+4\lambda\Bigg[3[\phi^2(x)-\Gamma_2^{-1}(x,x)]\delta(x,y)&+\int d^n v\, d^n u\,\Gamma_2^{-1}(x,v)\Gamma_2^{-1}(x,u)\\ \times\Big[3\Gamma_3(v,u,y)\phi(x)&+3\int d^n w\, d^n q\, d^n z\,\Gamma_2^{-1}(u,q)\Gamma_3(v,u,y)\Gamma_2^{-1}(x,z)\Gamma_3(w,q,z)\\ &-\int d^n w\,\Gamma_2^{-1}(x,w)\Gamma_4(v,u,w,y)\Big]\Bigg]=-\Gamma_2(x,y)\ ,\end{aligned} \tag{11}$$

$\cdots$.

Higher equations are very complicated; therefore, we do not show them.

Since it is not possible to solve exactly SD equations for an interacting theory, the best one can do is to use approximation schemes, which approach a solution in a systematic way. One should note that the same scheme applied to the SD hierarchy, (i) for all Green's functions, (ii) for connected Green's functions, and (iii) for proper vertices, would result in three different approximation series for a physical quantity, if expressed in terms of the corresponding Green's functions: G_k, W_k, or Γ_k. We can expect the fastest convergence in the case (iii), since proper vertices Γ_k are more fundamental objects, from which the functions W_k and G_k can be built. We consider here a truncation scheme,[2] where the Nth-order approximation for Green's functions is obtained by solving SD equations with the k-point Green's functions for $k>N$ set to zero. In the cases (i)–(iii) we shall study the approximation series for the propagator, i.e., the 2-point Green's function G_2, related to W_2 and Γ_2 by the formula

$$G_2=W_2=-\Gamma_2^{-1}\ . \tag{12}$$

Numerical results will be discussed for QFT in zero space-time dimensions, when the generating functional $Z(J)$ becomes an ordinary function, given by the one-dimensional integral. The Green's functions are simply numbers, calculable by numerical integration. By reflection symmetry odd Green's functions vanish.

First we examine case (i). The functional equation for all Green's functions (4) in zero-dimensional space-time becomes the ordinary third-order differential equation

$$m^2Z'+4\lambda Z'''=JZ \ , \tag{13}$$

requiring three boundary conditions, e.g., $Z(0)=1$, $Z'(0)=G_1=0$ (corresponding to the vanishing odd Green's functions) and $Z''(0)=G_2$. With this choice both $Z(J)$ and Green's functions, being its derivatives, depend on an arbitrary value G_2.

We can also obtain $Z(J)$ as a Taylor series with coefficients G_k calculated from SD equations (5). These equations in zero dimensions become algebraic:

$$\begin{aligned} &m^2G_2+4\lambda G_4=1 \ , \\ &m^2G_4+4\lambda G_6=3G_2 \ , \\ &m^2G_6+4\lambda G_8=5G_4 \ , \\ &\dots \ , \end{aligned} \tag{14}$$

and determine all even Green's functions in terms of one of them. With Green's functions expressed by G_2 the Taylor series for $Z(J)$ will reproduce a general solution of (13) for an arbitrary G_2. This is not the case if the SD equations are solved by truncation. The Nth-order truncation determines all Green's functions, including G_2^N; therefore, the Taylor series for $Z(J)$ gives the solution of (13) for which the third boundary condition is specified by

$$Z''(0)=\lim_{N\to\infty} G_2^N \ .$$

The lowest truncation ($N=4$) results in $G_2^4=m^{-2}$, which is the same as a zero-order perturbation solution. Thus, truncating Eqs. (14) we approach that solution of (13), which has a weak-coupling expansion. This explains why, as has been shown before,[3] the truncation scheme for $m^2>0$ converges to the Euclidean vacuum functional

$$Z_1(J)=N\int_{-\infty}^{\infty}\frac{dx}{(2\pi)^{1/2}}\exp\left[-\frac{m^2}{2}x^2-\lambda x^4+Jx\right] \ , \tag{15}$$

whereas, for $m^2<0$ it converges to

$$Z_2(J)=N\int_{-\infty}^{\infty}\frac{dx}{(2\pi)^{1/2}}\exp\left[\frac{m^2}{2}x^2-\lambda x^4+iJx\right] \ . \tag{16}$$

Z_1 and Z_2 have just weak-coupling expansions in the corresponding ranges of m^2.

In Fig. 1, we present numerical results for the rescaled propagator $G_2\lambda^{1/2}$ as a function of the parameter $m^2\lambda^{-1/2}$. The numerical results for $Z_1''(0)$ and $Z_2''(0)$ (solid lines) are compared with the values of the propagator, obtained by solving the three lowest truncations of Eqs. (14) (dashed lines). The approximation series approaches $Z_1''(0)$ when $m^2>0$, $Z_2''(0)$ when $m^2<0$, and diverges for $m^2=0$. Thus, the truncation method for SD equations for all Green's functions reproduces $Z_1(J)$ for $m^2>0$, and $Z_2(J)$ for $m^2<0$.

Let us discuss now the case (ii) of connected Green's functions. The differential equation for these functions (7) in zero dimensions reads

$$m^2W'+4\lambda[W'''+3W''W'+(W')^3]=J \ , \tag{17}$$

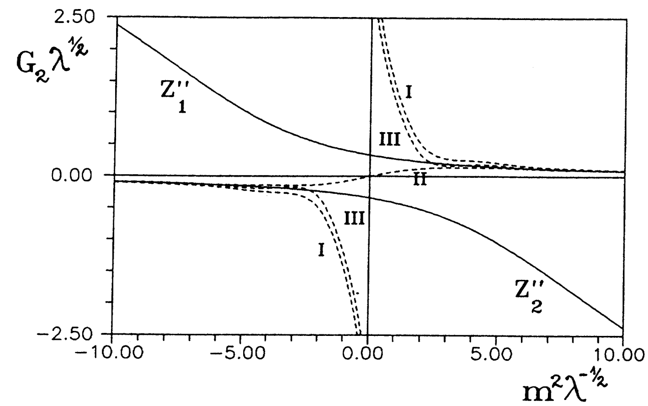

FIG. 1. The 2-point Green's function in zero-dimensional QFT plotted as a function of the parameter $m^2\lambda^{-1/2}$. The solid lines are the results calculated numerically. The dashed lines are the results of three lowest truncations of SD equations for all Green's functions.

and the hierarchy (8) simplifies to

$$\begin{aligned} &m^2W_2+4\lambda(W_4+3W_2^2)=1 \ , \\ &m^2W_4+4\lambda(W_6+12W_4W_2+6W_2^3)=0 \ , \\ &m^2W_6+4\lambda(W_8+18W_6W_2+30W_4^2+60W_2^2W_4)=0 \ , \\ &\dots \ . \end{aligned} \tag{18}$$

The lowest truncation ($W_4=0$) leads to a quadratic equation, which has two solutions:

$$W_2^{\pm}=\frac{-m^2\pm(m^2+48\lambda)^{1/2}}{24\lambda} \ . \tag{19}$$

W_2^+ is positive and closer to $Z_1''(0)$, and W_2^- is negative and closer to $Z_2''(0)$. A higher-order truncation gives an equation of higher degree for W_2. Choosing the roots closest to W_2^+ and W_2^- we obtain two series approaching $Z_1''(0)$ and $Z_2''(0)$, respectively. Since other Green's functions are determined by W_2, the Taylor series for $W(J)$ will reproduce $W_1=\ln Z_1$ with Z_1 given by (15) in the first case, and $W_2=\ln Z_2$ with Z_2 given by (16) in the second one.

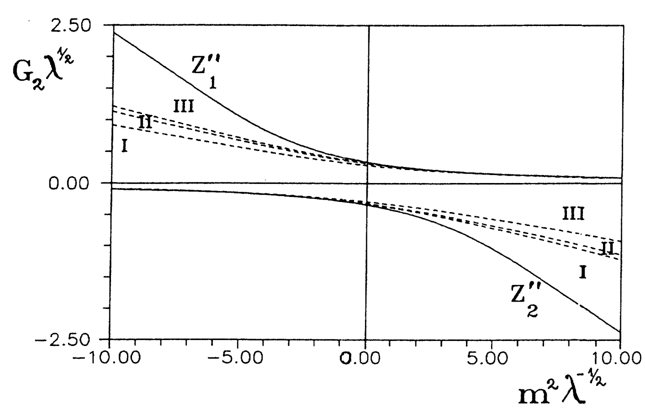

FIG. 2. As Fig. 1, but the dashed lines are the results of three lowest truncations of connected SD equations.

In Fig. 2 the results from the three lowest truncations of the connected SD equations (18) for $G_2\lambda^{1/2}$ (dashed lines) are compared with $Z_1''(0)$ and $Z_2''(0)$ (solid lines). In the whole range of $m^2\lambda^{-1/2}$ the positive-solution series approaches $Z_1''(0)$, but the convergence deteriorates as $m^2\lambda^{-1/2}$ becomes negative. The negative-solution series approaches $Z_2''(0)$; here, the convergence is bad for $m^2\lambda^{-1/2}>0$ and improves only when $m^2\lambda^{-1/2}<0$.

It is interesting to observe that the lowest truncation (19) of the connected SD equations is a Hartree approximation to QFT. The truncation scheme can be regarded as a way to go beyond the Hartree approximation. This is not the case for the truncation of SD equations for all Green's functions.

In case (iii), i.e., for proper vertices, the differential equation (10) in zero dimensions reads

$$m^2\phi+4\lambda[\phi^3+3(\Gamma'')^{-1}\phi+\Gamma'''(\Gamma'')^{-3}]=-\Gamma' , \tag{20}$$

and the hierarchy (11) is given by

$$\begin{aligned} &m^2-4\lambda(\Gamma_2^{-3}\Gamma_4+3\Gamma_2^{-1})=-\Gamma_2 ,\\ &\Gamma_4+4\lambda(6-\Gamma_2^{-3}\Gamma_6+9\Gamma_2^{-4}\Gamma_4^2+9\Gamma_2^{-2}\Gamma_4)=0 ,\\ &\Gamma_6-4\lambda(\Gamma_2^{-3}\Gamma_8-15\Gamma_2^{-2}\Gamma_6+90\Gamma_2^{-3}\Gamma_4^2\\ &\qquad -45\Gamma_2^{-4}\Gamma_4\Gamma_6+180\Gamma_2^{-5}\Gamma_4^3)=0 ,\\ &\cdots . \end{aligned} \tag{21}$$

The lowest truncation of (21) gives the same result for G_2, as in case (ii), i.e., for connected SD equations. The results of higher truncations are similar, although for case (iii) the convergence of the respective series is faster in the region where the approximation is good ($m^2\lambda^{-1/2}>0$ for positive solutions, and $m^2\lambda^{-1/2}<0$ for negative solutions). In this case an imaginary part appears in the solutions, when the convergence becomes bad.

To summarize, in zero-dimensional space-time the truncation scheme for SD equations for all Green's functions [case (i)] reproduces only that solution for $Z(J)$, which has a weak-coupling expansion. However, in general the truncation method is not related to the weak-coupling expansion. In case (ii) of connected Green's functions and (iii) of proper vertices the truncation method allows us to reproduce $Z_1(J)$ and $Z_2(J)$ in the whole range of m^2. As expected, the simpler Green's functions, the truncation of SD equations, give better results.

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