16 May 1996

PHYSICS LETTERS B

ELSEVIER

Physics Letters B 375 (1996) 213–216

Goldstone bosons in the Gaussian approximation

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Received 22 August 1995; revised manuscript received 9 February 1996
Editor: P.V. Landshoff

Abstract

The $O(N)$ symmetric scalar quantum field theory with $\lambda\Phi^4$ interaction is discussed in the Gaussian approximation. It is shown that the Goldstone theorem is fulfilled for arbitrary N.

1. Introduction

The theory of a real scalar field in n-dimensional Euclidean space-time with a classical action given by

$$S[\Phi] = \int [\tfrac{1}{2}\Phi(x)(-\partial^2 + m^2)\Phi(x) + \lambda(\Phi^2(x))^2]\, d^n x, \tag{1}$$

is the most mysterious part of the standard model. Although experimentally not observed, the scalar Higgs field with $m^2 < 0$ and internal $O(4)$ symmetry is necessary to give masses to interaction bosons in the Weinberg-Salam model of weak interactions without spoiling renormalizability. Moreover, the renormalized $\lambda\Phi^4$ theory has been almost rigorously proved [1] to be noninteracting, in contradiction to the perturbative renormalization, which can be performed order by order without any signal of triviality. Triviality shows up in the leading order of the $\frac{1}{N}$ expansion [2] in the $O(N)$ symmetric theory of N-component scalar field, $\Phi(x) = (\Phi_1(x), ..., \Phi_N(x))$. Other non-perturbative methods, like the Gaussian [3] and post-Gaussian [4,5] approximations, have been therefore applied to study renormalization of the scalar theory in the case when the number of field components is not large. However, a serious drawback of the Gaussian approximation for N-component field was the observation [6,7] that the Goldstone theorem seemed not to be respected exactly, only in the limit of $N \to \infty$ did the would-be Goldstone bosons become massless. Here we show that this statement is not true, and is due to a faulty interpretation. The Gaussian approximation of the $O(N)$ symmetric theory contains one massive particle and $(N-1)$ massless Goldstone bosons, in agreement with the exact result of the Goldstone theorem [8]. There was another claim of the existence of Goldstone bosons in the Gaussian approximation by Dmitrasinovic et al. [9], who found a pole in the four-point Green function, which

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PII S0370-2693(96)00198-0

they interpreted as a bound state of two massive elementary excitations in the theory. In our work we show that Goldstone bosons appear in the Gaussian approximation as elementary massless fields which are eigenvectors of the one-particle propagator matrix.

It is convenient to formulate the approximation method for the effective action, $\Gamma[\varphi]$, since all the Green's functions can be obtained in a consistent way, through differentiation of the approximate expression. The full (inverse) propagator, required for one-particle states analysis, is given by the second derivative of the effective action at $\varphi(x) = \phi_0$. The vacuum expectation value of the scalar field, ϕ_0, can be obtained as a stationary point of the effective potential,

$$V(\phi) = -\frac{1}{\int d^n x}\,\Gamma[\varphi]|_{\varphi(x)=\phi=\text{const.}}\,.$$

We shall calculate the effective action, using the optimized expansion (OE) [4]. The method consists in modifying the classical action of a scalar field (1) to the form

$$S_\epsilon[\Phi,G] = \int \tfrac{1}{2}\Phi(x)G^{-1}(x,y)\Phi(y)\,d^nxd^ny + \epsilon\left[\int \tfrac{1}{2}\Phi(x)[(-\partial^2+m^2)\delta(x-y) - G^{-1}(x,y)]\Phi(y)\,d^nxd^ny + \int \lambda(\Phi^2(x))^2\,d^nx\right], \tag{2}$$

with an arbitrary free propagator $G(x,y)$. The effective action, as a series in an artificial parameter ϵ, can be obtained as a sum of vacuum one-particle-irreducible diagrams with Feynman rules of the modified theory. The given order expression for the effective action is optimized, choosing $G(x,y)$ which fulfills the gap equation

$$\frac{\delta\Gamma_n}{\delta G^{-1}(x,y)} = 0, \tag{3}$$

to make the dependence on the unphysical field as weak as possible.

It has been shown that in the first order of the OE for a one-component field, the inverse of the free propagator can be taken in the form

$$\Gamma(x,y) = G^{-1}(x,y) = (-\partial^2 + \Omega^2(x))\delta(x-y), \tag{4}$$

and the Gaussian effective action (GEA) is obtained [4]. The effective potential, derived from the GEA for a constant background $\varphi(x) = \phi$, coincides with the Gaussian effective potential (GEP) [3], obtained before by applying the variational method with Gaussian trial functionals to the functional Schrödinger equation.

Here we shall calculate the effective action for an N-component field to the first order of the OE. In this case, the inverse of the trial propagator can be chosen in the form of a symmetric matrix

$$\begin{aligned}\Gamma_{i,i}(x,y) &= (-\partial^2 + M_i^2(x))\delta(x-y),\\ \Gamma_{i,j}(x,y) &= \Gamma_{j,i}(x,y) = M_{ij}^2(x)\delta(x-y),\end{aligned} \tag{5}$$

where the functions $M_i^2(x)$ and $M_{ij}^2(x)$ are variational parameters. The calculation of the effective action can be simplified, using the observation of Stevenson et al. [7] that for an $O(N)$ symmetric theory only the shift $\varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x))$ of the field sets a direction in the $O(N)$ space. Thus, the eigendirections of the free propagator matrix will be radial and transverse, and the variational parameters for the transverse fields should be equal, because of the remaining $O(N-1)$ symmetry. In the coordinate system in which the shift φ points in the $i = 1$ direction the (inverse) trial propagator can be chosen in the form of a diagonal matrix with

$$\begin{aligned}\Gamma_{11}(x,y) &= G^{-1}(x,y) = (-\partial^2 + \Omega^2(x))\delta(x-y),\\ \Gamma_{ii}(x,y) &= g^{-1}(x,y) = (-\partial^2 + \omega^2(x))\delta(x-y), \quad \text{for } i \neq 1,\end{aligned} \tag{6}$$

and the effective action in the first order of the OE is obtained in the form

$$\Gamma[\varphi] = -\int [\tfrac{1}{2}\varphi(x)(-\partial^2+m^2)\varphi(x) + \lambda(\varphi^2(x))^2]\,d^nx - \tfrac{1}{2}\,\mathrm{Tr\,Ln}\,G^{-1} - \frac{N-1}{2}\,\mathrm{Tr\,Ln}\,g^{-1}$$
$$+\tfrac{1}{2}\int (\Omega^2(x) - m^2 - 12\lambda\varphi^2(x))G(x,x)\,d^nx + \frac{N-1}{2}\int (\omega^2(x) - m^2 - 4\lambda\varphi^2(x))g(x,x)\,d^nx$$
$$-3\lambda\int G^2(x,x)\,d^nx - (N^2-1)\lambda\int g^2(x,x)\,d^nx - 2(N-1)\lambda\int G(x,x)g(x,x)\,d^nx. \qquad (7)$$

Requiring the effective action to be stationary with respect to small changes of variational parameters

$$\frac{\delta\Gamma}{\delta\Omega^2} = \frac{\delta\Gamma}{\delta\omega^2} = 0, \qquad (8)$$

results in the gap equations

$$\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 12\lambda G(x,x) - 4(N-1)\lambda g(x,x) = 0,$$
$$\omega^2(x) - m^2 - 4\lambda\varphi^2(x) - 4\lambda G(x,x) - 4(N+1)\lambda g(x,x) = 0, \qquad (9)$$

which determine the functionals $\Omega[\varphi]$ and $\omega[\varphi]$. When limited to a constant background $\phi = (\phi_1, ..., \phi_N)$, the the GEA for N-component field gives the effective potential

$$V(\phi) = \tfrac{1}{2}m^2\phi^2 + \lambda(\phi^2)^2 + I_1(\Omega) + (N-1)I_1(\omega) + \tfrac{1}{2}(m^2 - \Omega^2 + 12\lambda\phi^2)I_0(\Omega)$$
$$+ \frac{N-1}{2}(m^2 - \omega^2 + 4\lambda\phi^2)I_0(\omega) + 3\lambda I_0(\Omega)^2 + (N^2-1)\lambda I_0(\omega)^2 + 2(N-1)\lambda I_0(\Omega)I_0(\omega), \qquad (10)$$

with the functions $\Omega(\phi)$ and $\omega(\phi)$ determined by the algebraic equations

$$\Omega^2 - m^2 - 12\lambda\phi^2 - 12\lambda I_0(\Omega) - 4\lambda(N-1)I_0(\omega) = 0,$$
$$\omega^2 - m^2 - 4\lambda\phi^2 - 4\lambda I_0(\Omega) - 4(N+1)\lambda I_0(\omega) = 0, \qquad (11)$$

where

$$I_1(\Omega) = \tfrac{1}{2}\int \frac{d^np}{(2\pi)^n}\ln(p^2+\Omega^2),$$
$$I_0(\Omega) = \int \frac{d^np}{(2\pi)^n}\frac{1}{p^2+\Omega^2}. \qquad (12)$$

The same result for the $O(N)$ symmetric GEP was obtained before in the Schrödinger approach [7]. In the OE, a generalisation of the GEP to space-time dependent fields, the GEA (7), has been obtained. It enables us to derive not only the effective potential, but also one-particle-irreducible Green's functions at arbitrary external momenta in the Gaussian approximation.

The minimum of the GEP is at ϕ_0 fulfilling

$$\frac{\partial V}{\partial\phi_i} = (m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega))\phi_i = 0; \qquad (13)$$

therefore, in the unsymmetric minimum we have

$$B = m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega) = 0. \qquad (14)$$

In the GEP analysis for $N = 2$, it was pointed out by Brihaye and Consoli [6] that $\omega[\phi_0]$ is not equal to zero, which was interpreted as a violation of Goldstone theorem in the Gaussian approximation. For the same reason, Stevenson, Allès and Tarrach [7] admitted that also for general N the Gaussian approximation does not respect the Goldstone theorem. We would like to point out that this conclusion is unjustified, for Ω and ω are only variational parameters in the free propagator, and do not correspond to physical masses of scalar particles. The physical masses have to be determined as poles of the full propagator in the discussed approximation. The inverse of that propagator can be obtained as a second derivative of the GEA (7) with an implicit dependence, $\Omega^2[\varphi]$ and $\omega^2[\varphi]$, taken into account by differentiation of the gap equations (9). Upon performing the Fourier transform, the two-point vertex is calculated to be

$$\Gamma_{11}(p) = \left.\frac{\widehat{\delta^2\Gamma}}{\delta\varphi_1^2}\right|_{\varphi(x)=\phi_0} = p^2 + B + 8\lambda\phi_1^2 A(p),$$

$$\Gamma_{ii}(p) = \left.\frac{\widehat{\delta^2\Gamma}}{\delta\varphi_i^2}\right|_{\varphi(x)=\phi_0} = p^2 + B + 8\lambda\phi_i^2 A(p),$$

$$\Gamma_{ij}(p) = \Gamma_{ji}(p) = \left.\frac{\widehat{\delta^2\Gamma}}{\delta\varphi_i\delta\varphi_j}\right|_{\varphi(x)=\phi_0} = 8\lambda\phi_i\phi_j A(p), \tag{15}$$

where

$$A(p) = 1 - \frac{18\lambda I_{-1}(\Omega,p) + 2\lambda(N-1)I_{-1}(\omega,p) + 24\lambda^2(N+2)I_{-1}(\Omega,p)I_{-1}(\omega,p)}{1 + 6\lambda I_{-1}(\Omega,p) + 2\lambda(N+1)I_{-1}(\omega,p) + 8\lambda^2(N+2)I_{-1}(\Omega,p)I_{-1}(\omega,p)}. \tag{16}$$

and

$$I_{-1}(\Omega,p) = 2\int \frac{d^n q}{(2\pi)^n}\frac{1}{(q^2+\Omega^2)((p+q)^2+\Omega^2)}. \tag{17}$$

In the asymmetric state, with $\phi_1 = \phi_0$, $\phi_i = 0$ and $B = 0$, Eq. (15) is diagonal, with an inverse propagator $\gamma_1(p) = p^2 + 8\lambda A(p)\phi_0^2$, which corresponds to a massive particle, and $(N-1)$ inverse propagators $\gamma_i(p) = p^2$ of Goldstone bosons. Therefore, for any N the Gaussian approximation of the $O(N)$ symmetric theory does fully respect Goldstone theorem at the unrenormalized level.

I would like to acknowledge fruitful discussions with P. Stevenson and M. Consoli and the warm hospitality of Instituto Nazionale di Fisica Nucleare in Catania.

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